A First Course in Undergraduate Complex Analysis

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To the Instructor

These notes are intended for undergraduate mathematics students of modest background. A standard calculus sequence and some experience with proofs should be sufficient. A course in real analysis is not presumed, but I’ve included a short introduction to such strict epsilon-delta proofs so that students have some familiarity with them. These concepts are utilized at a few points in the course. You may tailor your course to avoid fundamental limit concepts from real analysis by assuming results requiring these ideas in the proof. On the other hand, you could emphasize the “learning to prove” part of this course by requiring proofs of these results. Note that exercises in the notes are for the most part meant to be specific examples. So, if you emphasize your course as “learning to prove”, then you could stay on a reasonable schedule by eliminating many of the exercises. In fact, by removing many or all of the exercises and by incorporating a little more topology in the plane, these notes could be used as a rigorous foundational course in complex analysis.

There are several ways of presenting the content for a complex analysis course. In these notes, I define an analytic function before series. Some authors prefer to introduce series earlier in order to use series results sooner. I chose the former presentation simply because undergraduate students seem to understand the material in this order more easily. However, an instructor could move the series chapter earlier with some small changes in the notes. This might make sense if, for example, you were interested in a rigorous foundational course.

For the sake of time I regretfully did not introduce trigonometric functions or hyperbolic functions. These are not critical to the development of the material but they are helpful to include for more examples. Many students are able to learn these easily on their own. In addition, this course does not emphasize applications of complex analysis. Also, some of the algebraic proofs are simple and can be tedious. You will need to decide whether your students will work on these easier problems to give them confidence (usually a good idea). Or, if you find them bored soon by these proofs, then speed the course up by omitting some of these proofs. The last couple of sections introduce some applications of residue theory. These are not essential, but provide some motivation to students for the material.

For the most part, the course followed a modified Moore method. However, when starting a new section which had several new definitions or concepts, typically we would work through several examples together in class to ensure students had a good understanding. In the endnotes you will see “together in class” to indicate this. Normally we would do these in the last half or quarter hour of the class period. Also, after watching the students present a thread of related problems, if I felt they did not understand some important concepts, I might give them something to try in that same class rather than waiting until the next class period. This is one of the great benefits of having students present their work. You can see immediately what issues they may be having.

The class for which these notes were used was unusually small with only three students. This meant the students had an extraordinary load in working through these notes.
themselves. An instructor with more students to spread the workload out may be able to include more material. There would also be more opportunity for students to learn from each other. For example, there were times when this small group did not catch a mistake with a proof or a solved problem in which case I would have to ask questions until they would see there was an issue.

The students were expected to present their fair share of the proofs and problems. In addition, I collected most of the problems from them, whether they presented those or not. I would not necessarily collect or grade basic practice exercises that were straightforward. I would grade both presentations and homework, but generously with an eye toward aiding their understanding. In other words, students received most of the points if they had thought seriously about the problems. Along with the presentations, I did include midterms and final exams in their final grade. The reasoning was to provide the students with a familiar assessment but more importantly to give them an incentive to review previous material periodically.

Finally, I am very willing to provide the notes in Latex to an instructor, as long as credit is given. Also, throughout the notes I have provided comments to the instructor specific to the topic at that point in the development. These are provided as endnotes in the annotation environment so that you may hide these in the student version of the notes.
Chapter 1

Complex Numbers and their Algebraic Properties

Throughout this course, you may assume all properties of the real numbers, including algebraic and analytical properties.

1.1 Basic Algebraic Properties

Definition 1. Define addition (+) of ordered pairs of real numbers \((x_1,y_1)\) and \((x_2,y_2)\) by 
\[(x_1 + x_2, y_1 + y_2)\].

Exercise 2. Choose any two pairs of real numbers and demonstrate the definition of addition.

Before looking further, do this:

Problem 3. 1 If two pairs of real numbers \((x_1,y_1)\) and \((x_2,y_2)\) are on the unit circle, geometrically define their product to be that point on the unit circle \((x_3,y_3)\) that is the sum of their angles, written as \((x_3,y_3) = (x_1,y_1)(x_2,y_2)\). Determine how \(x_3\) and \(y_3\) are related to \(x_1, y_1, x_2,\) and \(y_2\).

Definition 4. Define multiplication of any ordered pairs of numbers \((x_1,y_1)\) and \((x_2,y_2)\) (not just those on the unit circle) to be the ordered pair 
\[(x_1x_2 − y_1y_2, y_1x_2 + x_1y_2)\]
denoted \((x_1,y_1)(x_2,y_2)\).

Exercise 5. Compute the following:
\[
\begin{align*}
a. \quad & (2,3) + (-5,1/2), & b. \quad & (2,3)(-5,1) \\
& (2,-4) + (1/3,1), & d. \quad & (-1,-3)(-5,-2)
\end{align*}
\]

Theorem 6. \(\forall x, y \in \mathbb{R}\) (for all \(x\) and \(y\) in \(\mathbb{R}\))
\[
\begin{align*}
a. \quad & (1,0)(0,y) = (0,y), & b. \quad & (1,0)(x,0) = (x,0) \\
c. \quad & (0,1)(x,0) = (0,x), & c. \quad & (0,1)(0,y) = (-y,0)
\end{align*}
\]

Notation 7. We often use \(z\) for the pair \((x,y)\) where \(x\) and \(y\) are any real numbers. Thus, \(z := (x,y)\), where := means "is defined by".
Definition 8. Define the complex numbers, denoted \( \mathbb{C} \), to be the set of ordered pairs of real numbers with addition and multiplication defined as above. If \( z := (x, y) \) where \( x \) and \( y \) are real numbers, define the real part of \( z \) to be \( x \) and the imaginary part of \( z \) to be \( y \), denoted \( \text{Re}(z) \) and \( \text{Im}(z) \), respectively.

Definition 9. We call \((1,0)\) the multiplicative identity and \((0,0)\) the additive identity since the identities just return the complex number operating on them just as \(1\) and \(0\) do in the real numbers.

Assume the commutative, associative, and distributive properties of the real numbers to prove the next four theorems:

Theorem 10 (Commutativity). If \( z_1 \) and \( z_2 \) are complex numbers, then \( z_1 + z_2 = z_2 + z_1 \) and \( z_1z_2 = z_2z_1 \).

Theorem 11 (Associativity of Addition). If \( z_1, z_2, \) and \( z_3 \) are complex numbers, then \( (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \).

Theorem 12 (Distributivity). If \( z_1, z_2, \) and \( z_3 \) are complex numbers, then \( z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \).

Theorem 13 (Associativity of Multiplication). If \( z_1, z_2, \) and \( z_3 \) are complex numbers, then \( (z_1z_2)z_3 = z_1(z_2z_3) \).

Problem 14. Explain why the associative theorems allow us to write \( z_1 + z_2 + z_3 \) and \( z_1z_2z_3 \) unambiguously.

Definition 15. Define \( i := (0, 1) \).

Notation 16. We will use the following convention: for any real number \( x \), the complex number \((x, 0)\) will also be written as \( x \). Thus, when we are referring to complex numbers, \( 0 \) is really \((0,0)\) and \( 1 \) is really \((1,0)\).

Theorem 17. If \( z := (x,y) \), then \( z = (x,0) + (0,y) = x + iy \).

Another way to say this result is \( z \) may be represented by \( x + iy \).

Notation 18. Common practice is to use the letters \( z \) and \( w \) as complex numbers and \( x, y, u, \) and \( v \) as real numbers. Usually, the convention is to relate them by \( z := x + iy \) and \( w := u + iv \), but there may be exceptions to this convention.

We now want to show this definition of \( i \) is useful.

Theorem 19. \( i^2 = -1 \), where of course \( i^2 := ii \).

Problem 20. Suppose \( z_1 := x_1 + iy_1 \) and \( z_2 := x_2 + iy_2 \). First, formally add \( z_1 \) and \( z_2 \). ("Formally" means add “like” terms, those with “\( i \)” and those without “\( i \)”.) Next use the ordered pair definitions (1) to show that you arrive at the same answer.

Problem 21. Suppose \( z_1 := x_1 + iy_1 \) and \( z_2 := x_2 + iy_2 \). Formally multiply \( z_1 \) and \( z_2 \). ("Formally" means apply the standard FOIL method for multiplying \((a+b)(c+d)\), using \( i^2 = -1 \). Now use the ordered pair definition (4) for multiplication to show that you arrive at the same answer.

These last two problems show that the ordered pair notation and the notation using \( i \) are algebraically equivalent.
Exercise 22.

1. Compute the following:
   a. $\sqrt{3} - 4i + i(4 + \sqrt{3})$
   b. $(-3 + i)(1 - 5i)$
   c. $-3 - 4i - i(6 - i2)$
   d. $(-3 - i)(-1 - 5i)$

2. Locate the complex numbers $2 - 3i$ and $(-4, 3)$ in the plane.

Theorem 23. For all complex numbers $z$, $\text{Re}(iz) = -\text{Im}(z)$ and $\text{Im}(iz) = \text{Re}(z)$.

Problem 24. Show

1. $z_1 z_2 = 0$ implies either $z_1 = 0$ or $z_2 = 0$.
2. $z_1 \neq 0$ and $z_2 \neq 0$ imply $z_1 z_2 \neq 0$.

Definition 25. Let $z \in \mathbb{C}$. The additive inverse of $z$ is defined to be that complex number denoted $-z$ such that $z + (-z) = 0$.

Theorem 26 (Additive Inverse Theorem). For any $z \in \mathbb{C},$

1. The additive inverse of $z$ exists; and
2. The additive inverse is unique (meaning there is only one additive inverse to $z$).

Problem 27. Is $-(iy) = (-i)y = i(-y)$? Explain.

Note that this last problem shows that $-iy$ is unambiguous.

Definition 28. For any complex numbers $z$ and $w$, subtraction by a nonzero complex number $w$ from $z$ is defined by $z + (-w)$, denoted by $z - w$.

Problem 29. Solve the equation $z^2 + z + 1 = 0$ using ordered pair notation or $x + iy$ notation. Show this gives the same answer as if you applied the quadratic formula to the equation.

Definition 30. The multiplicative inverse of $z \in \mathbb{C}$, $z \neq 0$, is defined to be that complex number denoted $z^{-1}$ such that $z(z^{-1}) = 1$.

Theorem 31 (Multiplicative Inverse Theorem). Let $z := x + iy$ not equal to 0. Then

1. The multiplicative inverse of $z$ exists and is given by

$$z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

2. The multiplicative inverse of $z$ is unique.

Exercise 32. Determine the multiplicative inverse of $z := 2 + 3i$.

Definition 33. Division of a complex number $z$ by a nonzero complex number $w$ is defined to be $z(w^{-1})$, denoted $\frac{z}{w}$.

Note that $1/w$ is defined by setting $z = 1$ in this last definition.
Theorem 34. For all complex numbers \( w \) and \( z \), \( w \neq 0 \),

\[
\left( \frac{1}{w} \right) z = \frac{z}{w}
\]

and for all non-zero complex numbers \( w \) and \( z \),

\[
\left( \frac{1}{z} \right) \left( \frac{1}{w} \right) = \frac{1}{zw}
\]

We can use the following definition to simplify working with inverses.

Definition 35. Define the complex conjugate of \( z \) := \( x + iy \) by \( x - iy \), denoted \( \overline{z} \).

Exercise 36. Show

1. \( \overline{z - 5i} = z + 5i \).
2. \( \overline{iz} = -iz \).

Theorem 37. \( \forall z, z_1, z_2 \in \mathbb{C} \),

1. \( \overline{z} = z \).
2. \( \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \).
3. \( \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \).
4. \( \overline{z} = (x^2 + y^2) + 0i \). (You actually showed this earlier. Do you know where?)

Remark 38. By notation 16 the last statement of the last theorem may be expressed as \( \overline{z} \overline{z} = x^2 + y^2 \).

Problem 39. Show that the expression (1.1) for the inverse of \( z \) can be found by calculating:

\[
z^{-1} = \frac{\overline{z}}{\overline{z^2}}
\]

Be sure to justify each step with a definition or theorem.

Theorem 40. If \( z_1 := x_1 + iy_1 \) and \( z_2 := x_2 + iy_2 \), then

\[
\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} i
\]

where \( z_2 \neq 0 \).

Exercise 41. Express in the form \( a + bi \), where \( a \) and \( b \) are real numbers:

\[
\frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i}
\]

Problem 42. This is a writing assignment. How are complex numbers and ordered pairs of numbers in the plane similar and different?

Remark 43. We have now developed a type of algebraic system, a field, called the complex numbers, for the two-dimensional plane with four operations (addition, subtraction, multiplication, division) between pairs of points in the plane, as well as a unitary operation (complex conjugation) on a single point in the plane.
1.2 Moduli

Note that in problem (39) we encountered the expression \( z^* = x^2 + y^2 \) (by remark 38). Geometry motivates the following definition.

**Definition 44.** If \( z = x + iy \), then the modulus of \( z \) is defined by \( \sqrt{x^2 + y^2} \), denoted \( |z| \).

**Problem 45.** Discuss and write about:

1. Based on the definition, explain what \( |z| \) means geometrically. In other words, what is the geometric motivation mentioned above?
2. Based on the definition, explain what \( |z_1 - z_2| \) means geometrically.
3. Also explain the meanings, if any, of \( |z_1| < |z_2| \) and \( z_1 < z_2 \). Compare to the use of inequalities on the real line \( \mathbb{R} \).

**Exercise 46.**

1. Locate \( z_1 + z_2 \) and \( z_1 - z_2 \) geometrically (vectorially) when \( z_1 = 1 + 2i \) and \( z_2 = -3 + 3i \).
2. Sketch the graph of \( \{z : |z - 2 + i| = 3\} \).
3. Sketch the graph of \( \{z : |z - 3i| \leq 2\} \).
4. Sketch the graph of \( \{z : |z - 3i| \geq 2\} \).
5. Use geometric reasoning to determine the graph of \( \{z : |z - 1| = |z + i|\} \).

**Theorem 47.** For all \( z, z_1, z_2 \in \mathbb{C} \),

1. \( z^* = |z|^2 \).
2. \( |z^*| = |z| \)
3. \( \frac{z_1}{z_2} = \frac{z_1^*}{z_2^*} \)
4. \( |z_1 z_2| = |z_1||z_2| \)
5. \( \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|} \)

**Theorem 48.** For all complex numbers \( z \),

\[ Re(z) = \frac{z + z^*}{2}, \quad Im(z) = \frac{z - z^*}{2i} \]

**Problem 49.** Show that \( z \) is real if and only if \( z = z^* \).

Before looking at the next theorem, do the following:

**Problem 50.** Show that \( |1 + z + z^2| \leq 3 \) when \( |z| \leq 1 \). Attempt a brief calculation at showing this. Then conjecture a property that, if true, would make this problem much easier and provide an explanation as to why it is true.

To finish the last problem, prove the following theorem by first proving the lemmas following the theorem.
Theorem 51 (Triangle Inequality). \( \forall z_1, z_2 \in \mathbb{C}, |z_1 + z_2| \leq |z_1| + |z_2|. \)

Lemma 52.
\[
|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + z_2\overline{z_1}) + z_2\overline{z_2}
\]

Lemma 53. \( z_1\overline{z_2} + z_2\overline{z_1} = 2\Re(z_1\overline{z_2}) \leq 2|z_1||z_2| \)

Lemma 54. \( |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \)

Now use the Triangle Inequality to prove problem (50).

You may want to wait to do the next two until you finish problem 50.

Theorem 55. For all \( z \in \mathbb{C}, |\Re(z)| \leq |z| \).

Problem 56. Show that \( |\Re(2 + z + z^3)| \leq 4 \) when \( |z| \leq 1 \).

### 1.3 Polar and Exponential Forms

Recall the polar form of points in the plane from calculus.

Notation 57. For any complex number \( z \), the modulus \( |z| \) is also denoted \( r(z) \).

Problem 58. For any \( z \neq 0 \), show \( |z/r(z)| = 1 \). So geometrically, where in the plane does \( z/r(z) \) lie? What can you say about \( |x| \) and \( |y| \) relative to \( r(z) \)? Why?

Definition 59. Let \( z := x + iy \) be a non-zero complex number. Define the argument \( \theta(z) \) of \( z \) such that \( \cos[\theta(z)] = x/r(z) \) and \( \sin[\theta(z)] = y/r(z) \).

Problem 60. Does the definition of \( \theta(z) \) make sense (is it well-defined)? In other words, is there a non-zero complex number where there is no \( \theta(z) \) that satisfies the definition? Why or why not?

Notation 61. If the \( z \) is clear from the context, we will just use \( r \) instead of \( r(z) \) and \( \theta \) instead of \( \theta(z) \). In fact \( r(z) \) and \( \theta(z) \) imply we are dealing with functions here, which we will define more precisely in chapter 2. We also sometimes use the notation \( \arg z \) for \( \theta(z) \).

Problem 62. Show that any \( z \in \mathbb{C} \) can be expressed in the form \( r\cos \theta + ir\sin \theta \).

Definition 63. We define \( r\cos \theta + ir\sin \theta \) to be the polar form of \( z \).

Exercise 64. Determine \( r \) and \( \arg z \) for \( 1 + 2i, \ 1 - 2i, \ -1 + 2i, \ -1 - 2i, \ -1 \).

Exercise 65. Let \( z := 1 + 2i \). Graphically map \( z \) in the \( x-y \) plane to the \( r-\theta \) plane (where \( r \) is the first coordinate axis and \( \theta \) is the second coordinate axis).

Problem 66. Show that for any \( z \neq 0 \), \( r \) is unique but \( \theta \) is not unique.

Definition 67. Define \( \Theta \), called the Principal Value of \( \arg z \), to be that unique value of \( \theta \) such that \( -\pi < \theta \leq \pi \). We also use the notation \( \text{Arg} z \) for this unique choice of \( \theta \).

Exercise 68. Determine \( \text{Arg} z \) for \( 1 + 2i, \ 1 - 2i, \ -1 + 2i, \ -1 - 2i, \ -1 \). Compare to exercise (64).
Problem 69. Now generalize the results of the last exercise to any $z \neq 0$ and mathematically describe the relationship between $\arg z$ and $\text{Arg} z$ (or $\theta$ and $\Theta$).

Definition 70 (Euler’s formula). Define the following exponential notation.

$$e^{i\theta} := \cos \theta + i \sin \theta$$

Exercise 71. Show that, by problem 62, we may now express or represent $z$ in the form $re^{i\theta}$.

Definition 72. We define $re^{i\theta}$ to be the exponential form of $z$.

Exercise 73.

1. Express $z := 1/\sqrt{2} + i/\sqrt{2}$ in polar form and then in exponential form.
2. Express $z := 3/\sqrt{2} + 3i/\sqrt{2}$ in exponential form.
3. Express $z := -4 - 4\sqrt{3}i$ in exponential form.

Problem 74. Show that $|e^{i\theta}| = 1$ and $e^{i\theta} = e^{-i\theta}$.

Theorem 75 (Exponentiation Rules). Let $z_1 := r_1e^{i\theta_1}$, $z_2 := r_2e^{i\theta_2}$, and $z := re^{i\theta}$. Then

1. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$ and $z_1z_2 = r_1r_2e^{i(\theta_1 + \theta_2)}$.
2. \[ \frac{1}{z} = \frac{1}{r}e^{-i\theta}. \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)} \]

Exercise 76. Do the Exponentiation Rules or your proof give you a way to help you remember trig identities?

Problem 77. Compare these complex exponentiation rules to exponentiation rules for the reals. Are the complex exponentiation rules consistent with the reals? Do they make sense?

Exercise 78. Multiply $z_1 := -2 + 2i$ and $z_2 := 3 - 1i$ by first converting to exponential form and then using the last theorem.

By using this last theorem prove

Theorem 79. Let $z := re^{i\theta}$. Then for integers $n \geq 0$

$$z^n = r^n e^{in\theta}$$

This is also true for $n < 0$, which we will accept without proof:

Theorem 80. For integers $n < 0$

$$z^n = r^n e^{in\theta}$$

Exercise 81. Express $(1 + \sqrt{3}i)^{-10}$ in the form $a + bi$.

Theorem 82 (de Moivre’s Theorem). For $n \in \mathbb{Z}$

$$(\cos \theta + i\sin \theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Exercise 83.
1. Multiply \( z_1 := 1 + i \) and \( z_2 := 1 - i \) by first converting to exponential form and then using the last theorem. What is \( \arg(z_1z_2) \) ? Is it equal to \( \arg(z_1) + \arg(z_2) \)? (Recall \( \arg \) is multi-valued.)

2. Multiply \( z_1 := -1 + i \) and \( z_2 := -1 - i \) by first converting to exponential form and then using the last theorem. What is \( \arg(z_1z_2) \) ? Is it equal to \( \arg(z_1) + \arg(z_2) \)? (Recall \( \arg \) is multi-valued.)

Rely on the last exercise to explain this:

**Theorem 84** (arg theorem). For any complex numbers \( z_1 \) and \( z_2 \),

1. \( \arg(z_1z_2) = \arg z_1 + \arg z_2 \) but only if we interpret this equality in terms of sets. That is, if we pick any two of the values for the args above, then there is a value for the third arg where the addition is true.
2. \( \arg(z_1/z_2) = \arg z_1 - \arg z_2 \) but only if we interpret this in terms of sets.

**Problem 85.** Using an example of your choosing, graphically illustrate and describe in words what is meant by the arg theorem.

Before looking further on,

**Conjecture 86.** Trying some examples, conjecture if a similar result holds for \( \text{Arg} \).

**Exercise 87.** For \( z_1 := -1 + i \) and \( z_2 := i \), determine \( \text{Arg}(z_1z_2) \) and \( \text{Arg}z_1 + \text{Arg}z_2 \). Are they the same? What must we add or subtract from \( \text{Arg}(z_1z_2) \) to make them the same?

**Theorem 88** (Arg theorem). If \( \text{Re}(z_1) > 0 \) and \( \text{Re}(z_2) > 0 \), then \( \text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) \).

We will need this later:

**Theorem 89.** Two nonzero complex numbers \( z_1 := r_1e^{i\theta_1} \) and \( z_2 := r_2e^{i\theta_2} \) are equal if and only if \( r_1 = r_2 \) and there is a \( k \in \mathbb{Z} \) such that \( \theta_1 = \theta_2 + 2k\pi \).

**Exercise 90.** Suppose \( r_0 \) and \( \theta_0 \) are fixed constants (e.g. 5 and \( 3\pi/4 \)). Solve for \( z \) in \( z^2 = r_0e^{i\theta_0} \).
Chapter 2

Analytic Functions

2.1 Limits and Continuity

Recall what a function is in real analysis. A similar definition may be used in complex analysis, but we will define a function in a way probably different than you are familiar with.

Definition 91. A complex function $f$ is a set of ordered pairs of complex numbers $(z, w) \in \mathbb{C} \times \mathbb{C}$ such that for all $(z_1, w_1), (z_2, w_2) \in f$, $z_1 = z_2$ implies $w_1 = w_2$. Define the domain $D$ of $f$ and the range $R$ of $f$ by

$$D := \{ z \in \mathbb{C} : \text{there exists } w \in \mathbb{C} \text{ such that } (z, w) \in f \}$$

$$R := \{ w \in \mathbb{C} : \text{there exists } z \in \mathbb{C} \text{ such that } (z, w) \in f \}.$$

Henceforth, when we say function we mean complex function.

Exercise 92.

1. In previous math courses you were probably given a description in the following form: "A complex function $f$ is a set of complex numbers $D$, called the domain, along with a mapping (or rule) that assigns a unique complex number $w$ to each complex number $z$ in $D$. A particular mapping is written $w = f(z)$." Explain how this description of $f$ and $D$ are related to the definition given above.

2. Describe functions geometrically.

We will often specify a function by specifying the domain and the mapping as described in the last exercise. In fact, sometimes we will simply define the mapping, e.g. $f(z) = z^3 + 1$ without specifying the domain. Then we assume the domain of the function is all of $\mathbb{C}$ except for those values that don’t make sense:

Exercise 93.

1. Describe the domain of the following mappings:

(a) $$f(z) := \frac{1}{z^2 + 1}$$
Analytic Functions

(b) 
\[ f(z) := \frac{z}{z^2} \]

(c) 
\[ f(z) := \text{Arg}\left(\frac{1}{z}\right) \]

2. Given \( z := g(x, y) := x + iy \), express the mapping \( f(z) := z^2 + z + 1 \) in the form \( f(g(x, y)) = u(x, y) + iv(x, y) \). (This means write \( f \) in terms of \( x \) and \( y \).)

3. Write the function \( f(x, y) := x^2 - y^2 - 2y + i(2x - 2xy) \) in terms of \( z \), where \( z := x + iy \).

**Conjecture 94.** Can any mapping \( f(x, y) \) be written in terms of \( z \), e.g. such that \( f(x, y) = h(z) \) for some mapping \( h \)? If so, why? If not, which ones can or can’t be?

**Notation 95.** From now on, rather than using the cumbersome functional notation \( z := g(x, y) = x + iy \). \( f(g(x, y)) = u(x, y) + iv(x, y) \), we will just write \( f(z) = u(x, y) + iv(x, y) \) to mean expressing the mapping \( f \) in terms of \( x \) and \( y \).

**Exercise 96.** The concepts of the algebra of functions as you learned in calculus is the same (e.g. adding, multiplying, dividing, and composition of functions.) Describe the composition of two complex functions \( f \) and \( g \) graphically, verbally and in any other ways you find useful.

We will not overly emphasize fundamental analysis concepts, but you should have some exposure to them.

**Definition 97.** Let \( \varepsilon > 0 \). An \( \varepsilon \)-neighborhood of a point \( z_0 \) is the set given by \( \{z \in \mathbb{C} : |z - z_0| < \varepsilon\} \). A neighborhood of a point \( z_0 \) is just an \( \varepsilon \)-neighborhood of \( z_0 \) for some \( \varepsilon \).

**Exercise 98.** Describe what an \( \varepsilon \) neighborhood of a point \( z_0 \) is geometrically.

**Definition 99.** A set \( A \) in the \( z \)-plane is open if every point in \( A \) has a neighborhood which is a subset of \( A \).

**Problem 100.**

1. Give some examples of open sets and non-open sets in the \( z \)-plane. Do sets that have a "boundary" appear to be open sets?

2. Compare the definition of open to that of an open interval on the real line.

**Definition 101.** Let the \( f \) be a function whose domain \( D \) is open. Let \( z_0 \) and \( w_0 \) be two complex numbers, where \( z_0 \in D \). The limit of \( f(z) \) as \( z \) approaches \( z_0 \) is \( w_0 \) is defined by the following. For each positive number \( \varepsilon > 0 \) (e.g. \( \forall \varepsilon > 0 \)) there exists a positive number \( \delta \) (e.g. \( \exists \delta > 0 \)) such that

\[ |z - z_0| < \delta \text{ implies } |f(z) - w_0| < \varepsilon. \]

This statement defines the limit notation:

\[ \lim_{z \to z_0} f(z) = w_0. \]

Note that we do not specify that \( z_0 \) lie in the domain.
Exercise 102. Why did we include the assumption that the domain of \( f \) be open?

Problem 103. Suppose you want to show that \( \lim_{z \to 1+2i} (3z - i) = 3 + 5i \). (Why 3 + 5i?)

\[ f(z) := 3z - i, \quad w_0 := 3 + 5i \text{ and } z_0 := 1 + 2i. \]

1. Let’s suppose first we want to show that we can find a disc around \( z_0 \) so that \( f(z) \) is within 0.1 distance of \( w_0 \). (So here, \( \varepsilon = 0.1 \).)

   (a) Draw a picture in the \( w \) complex plane illustrating what points \( w \) are within 0.1 distance of \( w_0 \). So now you have a geometric picture.

   (b) Describe this geometric picture algebraically.

   (c) Determine what points in the \( z \) complex plane will guarantee that \( f(z) \) is within 0.1 distance of \( w_0 \). This is what we call \( \delta \). (Typically, we use the algebraic expression in (1b), and go backwards to get \( \delta \).)

   (d) Draw a picture with both the \( z \) plane and both the \( w \) plane indicating what you just found.

2. Do the same for \( \varepsilon = 0.01 \).

3. Does your \( \delta \) value depend on \( \varepsilon \)?

4. Do the same for \( \varepsilon = 1 \). (You may use the same \( \delta \) as the last question. Why?)

5. A proof that the limit exists means for any value of \( \varepsilon \) finding a \( \delta \) (which nearly always depends on \( \varepsilon \) ) that achieves this. Determine a \( \delta \) that guarantees that \( f(z) \) is within \( \varepsilon \) of \( w_0 \).

Exercise 104. Geometrically speaking, think about and discuss the ways \( z \) can be close to \( z_0 \).

Problem 105. Re-write the definition of limit of \( f(z) \), \( w_0 \), as \( z \) approaches \( z_0 \) using the concepts of \( \varepsilon \)-neighborhood and \( \delta \)-neighborhood instead. Explain the definition geometrically also.

Some basic practice with \( \varepsilon - \delta \) proofs:

Problem 106. Determine the following limits from the definition and prove their existence:

1. \( \lim_{z \to -3} (5z + 4i) \)
2. \( \lim_{z \to 2 + 4i} (5i) \)
3. \( \lim_{z \to 3i} z \)

Interpret each of these geometrically.

Rigorous \( \delta - \varepsilon \) proofs require careful analysis. As long as you can give plausible justification for the use of particular limits, we will allow the assumption that such limits exist.

We will assume the following theorem without proof:

Theorem 107. If the limit of a function exists, then that limit is unique.

Exercise 108. What do we mean when we say some object is “unique” in mathematics? What general approach would you go about showing the limit is unique in the last theorem?

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The contrapositive of the last theorem provides a way of showing when the limit of a function may not exist. It may be useful to employ sequences here. (Recall sequences are functions where the domain is the integers and the range lies in the complex numbers, similar to the definition of them in calculus.)

**Definition 109.** $z$ is the limit of an infinite sequence $z_1,z_2,z_3,...$ if $\forall \varepsilon > 0$, $\exists$ integer $N$ such that

$$n > N \text{ implies } |z_n - z| < \varepsilon$$

We then say the sequence converges to $z$ and use the notation $\lim_{n \to \infty} z_n = z$.

**Exercise 110.** Decide what the sequence $1 + \frac{1}{n}$ might converge to and then prove that it does.

**Problem 111.** Interpret this definition in words and geometrically.

**Problem 112.** Suppose the limit of $f(z)$ as $z$ approaches $z_0$, exists and call it $w_0$. Suppose a sequence $(a_n)$ converges to $z_0$. Then provide either a rigorous argument based on the definition of limit or a geometric argument explaining why $f(a_n)$ converges to $w_0$ as $n \to \infty$.

Suppose two sequences of points $(a_n)$ and $(b_n)$ in the domain both approach $z_0$, yet the sequences $(f(a_n))$ and $f(b_n)$ in the range do not approach the same number. Then by the last theorem the limit cannot exist.

**Problem 113.** Determine for which of the following the limit exists. If so, provide it. If not show why not.

1. $\lim_{z \to 0} \frac{z^2}{z}$
2. $\lim_{z \to 0} \frac{z}{z}$
3. $\lim_{z \to 0} \left( \frac{z}{z} \right)^2$

Hint: If you don’t think a limit exists, try sequences along the real axis, the imaginary axis, and another convenient line for each problem. More informally, you may think of a sequence as following a path along a line. For example, sequences whose range lie on the lines $x = 0$ or $y = x$.

The proof of the following theorem typically taught uses an $\varepsilon - \delta$ analysis approach, so we will assume this theorem is true:

**Theorem 114.** Suppose $f(z) := u(x,y) + iv(x,y)$, $z_0 := x_0 + iy_0$, and $w_0 := u_0 + iv_0$, where $z_0$ is in the domain of $f$. Then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

Prove the following by examining the meaning of the last theorem:

**Corollary 115.** Suppose $f(z) := u(x,y) + iv(x,y)$, $z_0 := x_0 + iy_0$, and $w_0 := u_0 + iv_0$, where $z_0$ is in the domain of $f$. Then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} \text{Re}(f(x,y)) = \text{Re}(\lim_{(x,y) \to (x_0,y_0)} f(x,y)) = \text{Re}(f(x_0, y_0)) = u_0$$
and

\[
\lim_{(x,y) \to (x_0,y_0)} \text{Im}(f(x,y)) = \text{Im}\left(\lim_{(x,y) \to (x_0,y_0)} f(x,y)\right) = \text{Im}(f(x_0,y_0)) = v_0
\]

Thus, what the last theorem and corollary say is that the limit of a complex function exists if and only if the usual limits of the two real functions, the real and imaginary parts, exist. Further, the limit of the real part of the complex function coincides with the real part of the limit of the complex function. Similarly for the imaginary part.

**Theorem 116.** Suppose that \( c \) is a complex number and

\[
\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0,
\]

where \( z_0 \) is in the domain of \( f \). Then

\[
\lim_{z \to z_0} [cf(z)] = cw_0
\]

\[
\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0,
\]

\[
\lim_{z \to z_0} [f(z)F(z)] = w_0W_0
\]

and if \( W_0 \neq 0 \),

\[
\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}
\]

**Corollary 117.** Let \( z_0 \) be a complex number. Then

1. \( \lim_{z \to z_0} z^n = z_0^n \) for \( n \in \mathbb{Z} \) and \( n \geq 0 \).
2. For \( z_0 \neq 0 \), \( \lim_{z \to z_0} z^n = z_0^n \) for \( n \in \mathbb{Z} \) and \( n < 0 \).
3. If \( P(z) \) is a polynomial in \( z \) then \( \lim_{z \to z_0} P(z) = P(z_0) \)
4. If \( P(z) \) and \( Q(z) \) are polynomials in \( z \) and \( Q(z_0) \neq 0 \), then \( \lim_{z \to z_0} P(z)/Q(z) = P(z_0)/Q(z_0) \)

Observe how we can build up more intricate functions from very simple functions by this process.

**Definition 118.** A function \( f \) is continuous at a point \( z_0 \) in the domain of \( f \) if \( f(z_0) \) exists and if, for each positive number \( \varepsilon > 0 \), there exists a positive number \( \delta \) (e.g. \( \exists \delta > 0 \)) such that

\[
|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon
\]

In \( \text{lim} \) notation this is written as

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

**Exercise 119.** Explain what this definition means geometrically. Also compare and contrast it with the definition of limit of a function.
Problem 120. Examples:

1. Create an example of a discontinuous complex function where \( \lim_{z \to z_0} f(z) \) exists and \( f(z_0) \) exists.

2. Create an example of a discontinuous complex function where \( \lim_{z \to z_0} f(z) \) exists and \( f(z_0) \) doesn’t exist.

3. Create an example of a discontinuous complex function where \( \lim_{z \to z_0} f(z) \) doesn’t exist and \( f(z_0) \) exists.

One proof of the following theorem is exactly analogous to what is done in real analysis using an \( \varepsilon - \delta \) proof. So we will assume this theorem to be true.

**Theorem 121.** Suppose \( f \) and \( g \) are continuous functions and that the range of \( g \) is contained in the domain of \( f \). Then \( f \circ g \) (composition of \( f \) and \( g \)) is continuous.

### 2.2 Differentiation

We will not use the following definition much, but knowledge of it will be helpful in examining derivatives. 11

**Definition 122.** Let \( f \) be a function where \( D \) is the domain of \( f \). The directional derivative of \( f \) at \( z \in D \) in the direction \( w \in \mathbb{C} \) is defined by

\[
\lim_{h \to 0} \frac{f(z + hw) - f(z)}{h}, \quad h \in \mathbb{R}
\]

when the limit exists. It is denoted by \( f'(z; w) \).

**Remark 123.** We say \( w \) is a direction by thinking of the ordered pair as a vector.

**Exercise 124.** Find the directional derivative of \( f(z) = z^2 \) in the direction \( w = 1 + i \).

**Definition 125.** Let \( f \) be a function where \( D \) is the domain of \( f \). The derivative of a function \( f \) at a point \( z_0 \in D \) is defined by

\[
\frac{df}{dz}(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},
\]

denoted \( f'(z_0) \). We also say \( f \) is differentiable at \( z_0 \).

**Problem 126.** Re-write the derivative of \( f \) at \( z_0 \) eliminating \( z \) using \( \Delta z := z - z_0 \). This is another common way of writing the definition.

**Problem 127.** Discuss what you think might be a geometric meaning of this definition of the derivative.

Recalling how we used sequences to show that limits do not exist, note that the limit above must exist and be the same no matter how \( z \) approaches \( z_0 \) "close" to \( z_0 \).

We will assume the following theorem without proof.

**Theorem 128.** If \( f \) is differentiable at \( z \), then the directional derivative exists at \( z \) in every direction.
**Remark 129.** Note the directional derivative differs from the derivative in that the limit in the derivative must exist and be the same in every direction and by any approach, e.g. along a straight line, a parabola (e.g. \( y = x^2 \)), or other smooth curve. Directional derivatives are taking the limits along straight lines.

The contrapositive of this last theorem can be quite useful.

**Problem 130.** Using the definition of derivative, determine if the derivative of the following functions exists (where the domains are \( \mathbb{C} \)) at all values of \( z \). If it does, determine the derivative for any \( z \). If it doesn’t, prove it does not for those values of \( z \) where it does not.

1. \( f(z) := z \)
2. \( f(z) := z^2 \)
3. \( f(z) := \text{Re}(z) \).
4. \( f(z) := \text{Im}(z) \).
5. \( f(z) := \overline{z} \)

**Problem 131.** Determine if the derivative exists at \( z = 0 \) for the following function

\[
f(z) := \begin{cases} \frac{z^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}
\]

**Theorem 132.** Suppose \( c \) is a complex constant and suppose the derivative of the complex function \( f \) exists at \( z \). Then

\[
\frac{d}{dz} c = 0 \quad \frac{d}{dz} z = 1 \quad \frac{d}{dz} (cf(z)) = cf'(z)
\]

The proofs of the following are the same as in calculus and real analysis. We will assume these are true here:

**Theorem 133.** Suppose \( c \) is a complex constant and suppose the derivatives of the complex functions \( f \) and \( g \) exist at \( z \). Then

1. Sums

\[
\frac{d}{dz} [f(z) + g(z)] = f'(z) + g'(z)
\]

2. Product rule

\[
\frac{d}{dz} [f(z)g(z)] = f'(z)g(z) + f(z)g'(z)
\]

3. Quotient rule. For those values of \( z \) where \( g(z) \neq 0 \),

\[
\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{f(z)g'(z) - f'(z)g(z)}{[g(z)]^2}
\]
4. Chain rule: Suppose that \( f \) has a derivative at \( z_0 \) and that \( g \) has a derivative at the point \( f(z_0) \). Then the function \( F(z) = g(f(z)) \) has a derivative at \( z_0 \), and
\[
F'(z_0) = g'(f(z_0))f'(z_0)
\]
A common way of writing this is to define \( w := f(z) \) and \( W = g(f(z)) \) and
\[
\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}
\]
One proof of the following theorem is the same as in calculus and real analysis. It is often shown using an induction argument for integers \( n \geq 0 \) and a small “trick” is often used for \( n < 0 \). We will assume this is true here:

**Theorem 134.** For integers \( n \),
\[
\frac{d}{dz} z^n = nz^{n-1}
\]
where \( z \neq 0 \) when \( n < 0 \).

**Exercise 135.** Determine \( f'(z) \) where
\[
f(z) := \frac{(1+z^2)^4}{z^2} \quad (z \neq 0)
\]

**Notation 136.** A subscript \( x \) or \( y \) on a variable means partial derivative with respect to \( x \) or \( y \). For example \( u_x \) means differentiate with respect to \( x \) keeping \( y \) constant, which also means the derivative of \( u \) along the line \( y \) constant.

**Problem 137.** Determine the derivatives \( u_x, u_y, v_x, \) and \( v_y \) for each of the functions. (As usual, \( u \) and \( v \) are the real and imaginary parts of \( f \).)

1. \( f(z) := z \)
2. \( f(z) := z^2 \)
3. \( f(z) := \text{Re}(z) \)
4. \( f(z) := i\text{Im}(z) \)

Do you notice a pattern when compared to which derivatives exist?

**Problem 138.** Suppose \( f(z) = u(x,y) + iv(x,y) \) and that \( f \) is differentiable at a point \( z_0 = x_0 + iy_0 \). Determine the directional derivative of \( f \)

1. along the line \( x = x_0 \), and
2. along the line \( y = y_0 \)
in terms of \( u \) and \( v \).

A consequence of problem (138) is the following theorem:

**Theorem 139 (Cauchy-Riemann Equations).** Suppose \( f'(z) \) exists at a point \( z_0 := x_0 + iy_0 \) in the domain of \( f \). Let \( f(z) := u(x,y) + iv(x,y) \). Then
1. The first order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$.

2. The first order partial derivatives of $u$ and $v$ satisfy the Cauchy-Riemann equations

\[ u_x = v_y, \quad u_y = -v_x \]

at $(x_0, y_0)$.

3. Also,

\[ f'(z_0) = u_x + iv_x = v_y - iu_y \]

at $(x_0, y_0)$.

**Exercise 140.** Using the contrapositive of the last theorem, determine where the following functions are not differentiable.

1. $f(z) = |z|^2$
2. $f(z) = \overline{z}$
3. $f(z) = 2x + ixy^2$
4. $f(z) = e^x e^{-iy}$
5. $f(z) = x^2 + iy^2$

The proof of the following theorem typically uses knowledge of Taylor’s series of functions of two real variables which is discussed in an advanced calculus course. We will assume the following theorem is true.

**Theorem 141 (Differentiability Sufficient Conditions).** Let the function $f(z) := u(x,y) + iv(x,y)$ be defined throughout some $\varepsilon$ neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that the first-order partial derivatives of the functions $u$ and $v$ with respect to $x$ and $y$ exist everywhere in that neighborhood. If those partial derivatives are continuous at $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations at $(x_0, y_0)$, then $f'(z_0)$ exists.

**Exercise 142.** Where are the following differentiable?

1. $f(z) := 1/z$
2. $f(z) := x^2 + iy^2$

**Problem 143.** Refer back to the function in problem 131. Determine the Cauchy-Riemann equations for this function at $z = 0$. Reconcile this result with what you found when you determined the differentiability of this function in that problem.

Remark: There are Cauchy-Riemann equations in polar form also. We will assume them without proof. (One way of proving them is to start with the Cauchy-Riemann equations in cartesian form and apply the chain rule.)

**Theorem 144 (Cauchy-Riemann Equations: Polar Form).** Suppose $f(z) := u(r, \theta) + iv(r, \theta)$, is defined in some neighborhood of a point $z_0 := r_0 e^{i\theta_0} \neq 0$. Also suppose the partial derivatives of $f$ with respect to $r_0$ and $\theta_0$ exist in this neighborhood and are continuous. Then the Cartesian Cauchy-Riemann equations

\[ u_x = v_y \quad u_y = -v_x \]

are equivalent to the polar Cauchy-Riemann equations

\[ ru_r = v_\theta \quad u_\theta = -rv_r \]
Thus, as long as the function $f$ is smooth enough, we can use the polar C-R equations to check for differentiability of $f$.

**Exercise 145.** Using the polar form of the Cauchy-Riemann equations, show that $f(z) := z^2$ is differentiable everywhere.

**Definition 146.** Analyticity:

1. A function of a complex variable $z$ is analytic (regular, holomorphic) in an open set if it has a derivative at each point in the set. (Note that here derivatives and analyticity are not defined on boundaries - only in neighborhoods.)

2. A function of a complex variable $z$ is analytic at a point $z_0$ if it is analytic in some open set containing $z_0$.

3. A function of a complex variable $z$ is entire if it is analytic in the entire plane.

4. If a function $f$ is not analytic at some point $z_0$ but is analytic at some point in every neighborhood of $z_0$, then $z_0$ is a singular point.

**Exercise 147.** Determine for what points, if any, for which the following functions are singular and for which the following functions are analytic:

1. $f(z) := 1/z$
2. $f(z) := |z|^2$
3. $f(z) := \frac{z^2 + 3}{(z - 3)^2(z^2 - 4z + 5)}$
4. $f(z) := e^z e^{i\pi}$

**Theorem 148.** Suppose $f$ and $g$ are analytic in a domain $D$ and $c$ is a complex constant. Then

1. $cf$,
2. $f \pm g$,
3. $fg$,
4. $f \circ g$

are analytic in $D$. Also, $f/g$ is analytic in $D$ excluding points where $g$ is zero.

**Problem 149.** Describe some general classes of functions that are entire.
Chapter 3

Some Elementary Complex Functions

We need some functions to use as examples in our studies and which are also useful in applied problems.

3.1 The Exponential Function

**Problem 150.** For the moment, let's use the symbol $\text{Exp}(z)$ to represent our complex exponential function on the complex numbers and $\exp(x)$ to be the usual exponential function on the reals. For aesthetic and practical reasons, we would like $\text{Exp}$ to satisfy the following properties:

- **Reduces to the reals correctly:** If we use a real number $x$ in for $z$, then we should have $\text{Exp}(x) = \exp(x)$.

- **Not trivial:** If $z$ is a non-real number, then $\text{Exp}(z) \neq \exp(x)$, where $z = x + iy$. In other words, we don’t want $\text{Exp}$ to be a trivial extension $\exp$.

- **Satisfies typical properties:** $\text{Exp}(z)$ should satisfy many of the properties we are familiar with for the reals only with complex numbers. If $z$ and $w$ are complex numbers, then we would like $\text{Exp}^z + w = \text{Exp}^z \text{Exp}^w$ and $\text{Exp}^z - w = \text{Exp}^z \text{Exp}^{-w}$.

Try defining a complex function on $\mathbb{C}$, calling it $\text{Exp}$, that satisfies the first property and see if your definition satisfies the rest or if you can modify it to satisfy the rest.

**Definition 151.** Define the complex exponential function by $e^x e^{iy}$ and denote it by $e^z$, where as usual $z := x + iy$. The domain is $\mathbb{C}$.

Note we have reverted to the typical notation $e^z$ from $\text{Exp}(z)$.

**Theorem 152.** For all $z, z_1, z_2 \in \mathbb{C}$,

1. $e^z$ reduces to the usual exponential function when $z$ is real.
2. $e^{z_1 + z_2} = e^{z_1} e^{z_2}$.
3. $e^{z_1 - z_2} = e^{z_1} e^{-z_2}$.
4. $\frac{d}{dz} e^z = e^z$. For what values of $z$ does this derivative exist?
5. $e^z \neq 0$ for any $z$. 
6. \(e^z\) is periodic with period \(2\pi i\). A function \(f(z)\) is periodic with period \(T\) if \(f(z + T) = f(z)\). (Note how this result differs from the usual exponential function.)

**Problem 153.** Solve \(e^z = -1\) for \(z\). (See exercise 90 to recall how we solved an earlier equation.)

**Problem 154.** Let \(z := x + iy\). Answer the following.

1. What is \(|e^z|\) in terms of \(x\) and \(y\)?

2. Show that \(|e^z|^2 \leq e^{2|z|}\). Also, explain in words the geometric consequences of this expression. In other words, what does this expression mean geometrically?

**Exercise 155.** Show that

\[e^{z + \pi} = \sqrt{e}(1 + i)\]

**Exercise 156.** Explain why the function \(5z^3 - ze^z - e^{-z}\) is entire.

### 3.2 The Logarithmic Function

**Problem 157.** \(^1\) When considering only real numbers \(x\) and \(y\), what is the solution of \(e^y = x\), solving for \(x\)?

Before looking further, do this problem:

**Problem 158.** Now we want to solve \(e^w = z\) where \(w\) and \(z\) are complex. Let \(z := re^{i\Theta}\). (Remember what \(\Theta\) is? See definition 67.) Also let \(w := u + iv\). Solve for \(u\) and \(v\).

We define the complex log as the solution in the last problem:

**Definition 159.** Let \(z := re^{i\Theta}\). The complex log of \(z\) is defined by \(\ln r + i(\Theta + 2n\pi)\) for \(n \in \mathbb{Z}\), \(z \neq 0\), and is denoted by \(\log z\).

Question: Does \(\log z\) have a single value?

**Problem 160.** Show that \(\log z = \ln|z| + i\arg z\) for \(n \in \mathbb{Z}\), \(z \neq 0\).

**Exercise 161.** By applying the definition, determine or simplify:

1. \(\log(-1 + \sqrt{3}i)\)
2. \(e^{\log(1+2i)}\)
3. \(\log(e^{1+2i})\)

**Theorem 162.**

1. \(e^{\log z} = z, z \neq 0\).
2. \(\log(e^{2}) = z + 2n\pi i\) for \(n \in \mathbb{Z}\) and for \(z \in \mathbb{C}\).

**Definition 163.** The principal value of \(\log z\) is defined to be \(\ln r + i\Theta, z \neq 0\), denoted by \(\text{Log} z\).
Problem 164. How is Log\(z\) related to log\(z\)? Write an equation relating them.

Exercise 165. By applying the definition, determine:

1. log\(1\) and Log\(1\)
2. log\((-1)\) and Log\((-1)\)
3. log\(i\) and Log\(i\)
4. log\((-i)\) and Log\((-i)\)
5. log\((1-i)\) and Log\((1-i)\)

complex Logs:

Problem 166. Show that Log\([(1+i)^2]\) = 2Log\((1+i)\) but Log\((-1+i)^2\) \(\neq\) 2Log\((-1+i)\).

Conjecture 167. Can you explain in general why the second one “didn’t work”? Can you conjecture under what conditions equality might hold?

Theorem 168. For \(r > 0\), \(-\pi < \Theta < \pi\), Log\(z\) is differentiable and analytic. (Recall one method we used to determine if a function is differentiable?)

Problem 169. Show that Log\(z\) is not differentiable at \((-1,0)\) by taking the difference quotient limit along the curve \(e^{i\theta}\) above the y-axis and by taking the difference quotient limit along the curve \(e^{i\theta}\) below the y-axis.

Notice that we used \(-\pi\) as the “dividing line” when restricting \(\Theta\) to \(\Theta\) (which makes up the definition of Log\(z\)). There is nothing special about \(-\pi\):

Problem 170. Let \(\alpha\) be any real number. Is the following function single-valued? What is the largest domain in which it is analytic?

\[\ln r + i\theta, \quad \text{Domain: } r > 0, \alpha < \theta \leq \alpha + 2\pi\]

Definition 171. An open set S is path connected if each pair of points in S can be connected by a polygonal line (e.g. a finite number of line segments connected end to end). A domain is an open set that is path connected.

Remark 172. Note that the word “domain” has two meanings. Here it is used to describe a type of set but previously it was used as the allowable values for the first coordinate of a function. This latter use of “domain” is always associated with a function while the former is just a type of set.

Exercise 173. Which of the following sets are path connected? domains?

1. The set given by the disk \(|z - 2| < 2\).
2. The set given by the disk \(|z - 2| \leq 2\).
3. The set given by the union of the disks \(|z - 2| < 1\) and \(|z + 1| < 1\).
4. The set given by \(r > 0, -\pi < \Theta < \pi\).

We will always discuss differentiability over open sets and domains. Only “one-sided” derivatives make sense when the set has a boundary. We will not need these here.
Definition 174. A branch of a multi-valued function $f$ is any single-valued function $F$ that is analytic in some domain of each point $z$ where $F(z) = f(z)$. The domain of $F$ is this domain. We call $\log z$ the Principal Branch of $\log z$. A branch cut is a portion of a line or curve introduced to make $f$ into the single valued function $F$.

For example, the branch cut to obtain $\log z$ is $\theta = \pi$.

We will accept the following theorem without proof.

Theorem 175. log properties: For $z_1 \neq 0$ and $z_2 \neq 0$,

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log(z_1 / z_2) = \log z_1 - \log z_2$

where we interpret this equality in terms of sets as in theorem 84.

Problem 176. By re-examining theorem 84 and exercise 83, explain what is meant by “where we interpret this equality in terms of sets ” in the last theorem.

3.3 The Square Root and Functions with Complex Exponent

Before looking at the next definition:

Problem 177. Represent $z \in \mathbb{C}$ in exponential notation and conjecture what $z^{\frac{1}{2}}$ might be.

Definition 178. Define the square root of $z$ by

$$e^{\frac{1}{2} \log z}, z \neq 0 \text{ and } 0^\frac{1}{2} := 0,$$

denoted $z^{\frac{1}{2}}$ or $\sqrt{z}$.

Problem 179. Is $z^{\frac{1}{2}}$ continuous at $z = 0$?

Problem 180. In general, is $\sqrt{z}$ single or multi-valued? If the latter, describe the possible values $\sqrt{z}$ can have in general.

Definition 181. The principal value of $\sqrt{z}$ is when $\Theta$ is used in place of $\theta$ in the definition of $z^{\frac{1}{2}}$, denoted $\text{PV} \sqrt{z}$.

Exercise 182. By applying the definition, determine these values as well as their principal values:

$$\sqrt{4}, \quad \sqrt{i}, \quad \sqrt{1+i}, \quad \sqrt{5(-1 - \sqrt{3}i)}$$

To define functions of any complex exponent, we just extend the definition used for square root:

Definition 183. Let $c$ be any complex constant. Define $z$ to the power $c$ by

$$e^{c \log z}, z \neq 0 \text{ and } 0^c := 0,$$

denoted $z^c$. 
Note that $c$ can have rational or irrational, real or complex components.

**Problem 184.** Is $z^{\frac{1}{4}}$ single or multi-valued? How many possible values can it have in general?

**Definition 185.** The principal value of $z^c$ is when $\log z$ is used in place of $\log z$, e.g.

$$e^{\log z}, z \neq 0 \text{ and } 0^c := 0.$$ denoted $\text{PV} z^c$.

Other branches could also be used, for example $r > 0$ and $\alpha < \theta < \alpha + 2\pi$.

**Exercise 186.** Determine

1. $(1 + i)^{\frac{1}{2}}$ and its principal value.
2. $(1 - \sqrt{3})^{\frac{1}{2}}$ and its principal value.

We will assume these standard results without proof:

**Theorem 187** (Algebra of Complex Exponents). Let $c$, $d$, and $z$ denote complex numbers with $z \neq 0$. Then $1/z^c = z^{-c}$, $(z^c)^n = z^{cn}$, $z^c z^d = z^{c+d}$, and $z^c/z^d = z^{c-d}$.

We will assume these results without proof:

**Theorem 188** (Analytic Properties of Complex Exponents). Let $c$ and $z$ denote complex numbers. Then

1. When a branch is chosen for $z^c$, then $z^c$ is analytic in the domain determined by that branch.

2. $$\frac{d}{dz}z^c = cz^{c-1} \text{ on the branch of } z^c$$
Chapter 4

Integrals

4.1 Derivatives and Integrals of Functions of One Real Parameter

Definition 189. Suppose $w$ is a complex function of a real parameter $t$ on an open set $D$ in $\mathbb{R}$. The derivative of $w$ with respect to $t \in D$ is defined by

$$u'(t) + iv'(t),$$

denoted $w'(t)$, where $u$ and $v$ are the real and imaginary parts, respectively, of $w$, and assuming the derivatives of $u$ and $v$ exist at $t$ in some open interval containing $t$. When $w'$ exists at $t$, we say it is differentiable at $t$.

Theorem 190. Let $z_0$ be a complex constant and $w$ be a differentiable complex function at $t$. Then

1. \[
\frac{d}{dt}(z_0w(t)) = z_0w'(t)
\]

2. \[
\frac{d}{dt}e^{zt} = z_0e^{zt}
\]

We will assume the following theorem. Its proof is similar to that in calculus.

Theorem 191. Suppose the complex functions $f(t)$ and $g(t)$ are differentiable with respect to its of a real parameter $t$ on an open set $D$ in $\mathbb{R}$. Then, at $t \in D$,

1. Sums

\[
\frac{d}{dt}[f(t) + g(t)] = f'(t) + g'(t)
\]

2. Product rule

\[
\frac{d}{dt}[f(t)g(t)] = f'(t)g(t) + f(t)g'(t)
\]
3. **Quotient rule.** For \( g(t) \neq 0 \),

\[
\frac{d}{dt} \left[ \frac{f(t)}{g(t)} \right] = \frac{f'(t)g(t) - f(t)g'(t)}{(g(t))^2}
\]

Notice you are not assuming the chain rule for complex functions. You will prove that as a theorem later.

**Problem 192.** Using the definition or the last theorem, show that

1. \( \frac{d}{dt} [w(-t)] = -w'(-t) \)

where \( w'(-t) \) means the derivative of \( w(t) \) with respect to \( t \) evaluated at \( -t \).

2. \( \frac{d}{dt} [w(t)]^2 = 2w(t)w'(t) \)

**Problem 193.** Let \( w(t) := e^{it} \) on the set \( 0 \leq t \leq 2\pi \). Is the mean value theorem as stated in calculus satisfied by \( w \)? Explain. Note that you may look up the mean value theorem in a calculus book. (Hint: Use the modulus.)

**Problem 194.** State the intermediate value theorem from calculus as applied to real functions. Might the intermediate value theorem apply to complex functions of a real parameter? Why or why not?

**Definition 195.** Let \( w(t) := u(t) + iv(t) \) be a complex function of a real parameter \( t \) on an interval \( a \leq t \leq b \), where \( u \) and \( v \) are integrable over this interval. Then the integral of \( w \) over \( a \leq t \leq b \), is defined by

\[
\int_a^b u(t)dt + i\int_a^b v(t)dt,
\]

denoted

\[
\int_a^b w(t)dt.
\]

If the interval is unbounded, then the integral of \( w \) is defined similarly except improper integrals of \( u \) and \( v \) are used. We say \( w \) is integrable if the integral exists and is unique.

**Definition 196.** A real function of a real parameter \( t \) is piecewise continuous on an interval \( a \leq t \leq b \) if it is continuous on \( a \leq t \leq b \) except for a finite number of discontinuities at which the one-sided limits exist. A complex function is piecewise continuous if its real and imaginary parts are piecewise continuous. (Informally, there are a finite number of jumps.)

You may use results from integration over the reals for any proofs or problems.

By reviewing how definite integration is defined in calculus (e.g. as Riemann integrals), using a geometric description of integration, justify (though not necessarily prove in a rigorous manner) the following theorem.

**Theorem 197.** If \( w \) is piecewise continuous on \( a \leq t \leq b \), then it is integrable on \( a \leq t \leq b \).
Theorem 198.

\[ \text{Re} \left( \int_a^b w(t)\,dt \right) = \int_a^b \text{Re}(w(t))\,dt \quad \text{and} \quad \text{Im} \left( \int_a^b w(t)\,dt \right) = \int_a^b \text{Im}(w(t))\,dt \]

Theorem 199 (Fundamental Theorem of Calculus for functions of a real parameter). Suppose that \( w(t) := u(t) + iv(t) \) is continuous and the derivative of \( W(t) := U(t) + iV(t) \) exists on \( a \leq t \leq b \). If \( W'(t) = w(t) \) on \( a \leq t \leq b \), then

\[ \int_a^b w(t)\,dt = W(t)|_a^b := W(b) - W(a) \]

Can you recall what \( W \) would be called in the case of real functions?

Exercise 200. Calculate

1. \[ \int_1^2 \left( \frac{1}{t} - i \right)^2 \,dt \]

2. \[ \int_1^\infty e^{-\alpha t} \,dt \]

where \( \text{Re}(\alpha) > 0 \).

3. \[ \int_{-\infty}^0 e^{-\beta t} \,dt \]

where \( \text{Re}(\beta) < 0 \).

Justify your steps.

Problem 201. Explain why the fundamental theorem of calculus is essential for determining either complex or real integrals.

Theorem 202. Suppose the complex functions \( f(t) \) and \( g(t) \) are integrable with respect to its real parameter at \( t \) over the interval \( a \leq t \leq b \). Suppose \( c \) is a complex constant. Then

1. \[ \int_a^b c f(t)\,dt = c \int_a^b f(t)\,dt \]

2. \[ \int_a^b [f(t) + g(t)]\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt \]

The following result is very important in Fourier series and the many applications that use Fourier series:
Problem 203. Let \( m, n \in \mathbb{Z} \). Show that
\[
\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 
0 & \text{if } m \neq n \\
2\pi & \text{if } m = n
\end{cases}
\]

Here is a taste of how to use complex variables to determine integrals of real functions.

Problem 204. Determine the two integrals
\[
\int_0^\pi e^x \cos x \, dx, \quad \int_0^\pi e^x \sin x \, dx
\]
by evaluating the integral
\[
\int_0^\pi e^{(1+i)x} \, dx
\]
and then using real and imaginary parts of the result.

Problem 205. This problem is to be done with real functions and real integrals. For a real function \( f \), in terms of Riemann sums explain why
\[
\left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt.
\]
Illustrate the plausibility of this by a reasonable example.

The following similar theorem is a very important theorem in analysis. You will prove this by a couple of lemmas following the statement of the theorem.

Theorem 206 (Bound on Moduli of Integrals). Suppose \( w \) is a complex function of a real parameter integrable on \( a \leq t \leq b \). Then
\[
\left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt
\]

If you aren’t able to prove a lemma, continue on to the next one and assume the previous lemma(s) are true. Then go back and try to finish proving the ones you were stuck on.

Lemma 207. Let \( w \) be as in the last theorem. Let \( c \) be the complex number \( c := \int_a^b w(t) \, dt \). Represent \( c \) by \( c := r_0 e^{i\theta_0} \), \( r_0 \neq 0 \). Then
\[
r_0 = \int_a^b e^{-i\theta_0} w(t) \, dt
\]

Now, with the same notation as the last lemma,

Lemma 208.
\[
r_0 = \int_a^b \text{Re}(e^{-i\theta_0} w(t)) \, dt
\]

Again, with the same notation,

Lemma 209.
\[
\text{Re}(e^{-i\theta_0} w) \leq |w|
\]
so using the last lemma
\[
r_0 \leq \int_a^b |w(t)| \, dt
\]
Now prove Theorem 206 with this lemma.

**Problem 210.** Suppose, for \( x \) satisfying \(-1 \leq x \leq 1\),

\[
P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left( x + i \sqrt{1 - x^2} \cos \theta \right)^n \, d\theta \quad \text{for any non-negative } n \geq 0.
\]

By using the last theorem and the triangle inequality, Theorem 51, show that \(|P_n(x)| \leq 1\) for any \( x \) in \(-1 \leq x \leq 1\).

Remark: The function \( P_n \) is called a *Legendre Polynomial* and arises in applications with circular symmetry.

### 4.2 Contours

In order to define integrals of complex functions \( f \) of complex variables, we need to build up a few concepts. Because the domain is in the plane, we define these integrals on curves in the plane. In this section, we define the type of curves we are interested in.

**Definition 211.** An *arc* is a continuous function with domain an interval \( a \leq t \leq b \) on the real line and whose range is in the complex plane. Thus, the mapping of an arc is defined by \( x(t) + iy(t) \) where \( x(t) \) and \( y(t) \) are continuous real mappings on \( a \leq t \leq b \). Arcs are often denoted by \( C \).

**Definition 212.** An arc \( C \) is a simple arc or *Jordan arc* if, on \( a \leq t \leq b \), \( t_1 \neq t_2 \) implies \( C(t_1) \neq C(t_2) \), e.g. the function is one-to-one.

**Definition 213.** An arc \( C \) is a simple closed curve if, on \( a < t < b \), \( t_1 \neq t_2 \) implies \( C(t_1) \neq C(t_2) \), and \( C(a) = C(b) \).

**Exercise 214.** For the mappings given, answer questions that follow.

1. \( C(t) := 3t + i4t^2 \) on \(-2 \leq t \leq 2\).
2. \[
C(t) := \begin{cases} 
1 + it & \text{if } 0 \leq t \leq 2 \\
(t-1) + 2i & \text{if } 2 \leq t \leq 3 
\end{cases}
\]
3. \[
C(t) := \begin{cases} 
1 + it & \text{if } 0 \leq t \leq 2 \\
t + 2i & \text{if } 2 \leq t \leq 3 
\end{cases}
\]
4. \( C(\theta) := z_0 + R_0 e^{i\pi \theta} \) on \( 0 \leq \theta \leq 2\).
5. \( C(\theta) := z_0 + R_0 e^{2\pi \theta} \) on \( 0 \leq \theta \leq 2\).
6. \( C(\theta) := z_0 + R_0 e^{2\pi \theta} \) on \( 0 \leq \theta \leq 1\).
7. \( C(\theta) := e^{-i\theta} \) on \( 0 \leq \theta \leq 2\pi\).
8. \( C(t) := 1 + it \) on \(-1 \leq t \leq 1\).
Questions:

1. Which of the mappings with the given domains are arcs, simple arcs, and/or simple closed curves?

2. What is the range of each? Answer by indicating what the range is geometrically in the z-plane.

3. What direction is followed? This means write an arrow on the range of the contour to indicate which direction one moves along the range as t increases.

4. Do any of the mappings traverse parts of the range more than once?

5. Which “follow the same path”? By “same path” we roughly mean the arcs have the same range and traverse that range in the same direction. In addition, if they traverse parts of the range more than once, then they would traverse it the same number of times.

Two different arcs may “follow the same path” in the plane. They seem equivalent in some sense even though the functions defining the arcs are technically different. 18

Problem 215. Interpret the meaning of a simple arc and a simple closed curve geometrically.

Problem 216. Suppose we have the arc defined by the mapping C(t) := t + it^2, −2 ≤ t ≤ 2. Suppose we change the independent variable from t to τ by the transformation τ = t/2. (This is called re-parameterization.)

1. Express the mapping and the domain for the arc in terms of the new variable τ. Note that this is a new arc since it is a different function.

2. Do they “follow the same path”?

Problem 217. Use the same arc as in problem 216 but instead try the transformation τ = t^2. Write the mapping and domain for the new arc in terms of the new variable τ. Did you encounter any issues? What do you notice that is qualitatively different from the last problem? Does this new arc have the “same path” as the original arc?

Lesson: One must be careful when transforming the independent variable.

Notation 218. Up to this point, we have been using the notation C(t) to denote the arc mappings. Sometimes it will be more suggestive to use the notation z(t) to denote the mapping of an arc C.

Definition 219. An arc z(t) := x(t) + iy(t) with domain D, is differentiable at t ∈ D if each of the components x and y is differentiable at t.

Definition 220. Suppose the arc C given by z(t) on a ≤ t ≤ b is differentiable on a < t < b. Then, if |z'(t)| is integrable then the length of the arc C is defined to be

\[ \int_a^b |z'(t)|\,dt \]

Problem 221. Approximate the integral above by a Riemann sum of n terms and the derivative by difference quotients, e.g. roughly

\[ \int_a^b |z'(t)|\,dt \approx \sum_{i=1}^{n} \left| \frac{z(x+i\Delta t) - z(x)}{\Delta t} \right| \Delta t \]
Integrals

From this, explain why the definition of the length is a sensible definition for the length of an arc. Simplifying the sum above would help, including explicitly examining $|\Delta z|$. Also perhaps drawing a picture would help too.

**Exercise 222.** Calculate the length of contour C and graph the range of C:

1. C given by $z(\theta) := e^{i\theta}$ on $0 \leq \theta \leq 2\pi$.
2. C given by $z(\theta) := e^{i\theta}$ on $0 \leq \theta \leq 4\pi$.
3. C given by $z(t) := 1 + it^2$ on $-2 \leq t \leq 2$.

**Problem 223.** In the last problem of the last exercise, let $\tau = t/2$. Calculate the integral by applying the transformation like you did for problem 216. Then integrate over $\tau$. Do you get the same number as in the last problem of the last exercise?

Conjecture: With proper conditions on the parameterization, the parameterization does not affect the length. (Consider what might happen if you let $\tau = t^2$ in the above. Would you obtain the same length then?)

Justify the following theorem by applying substitution. Also, try to provide reasons for the conditions on $\phi$ in this theorem.

**Theorem 224.** Let arc C have parameterization in t over $a \leq t \leq b$. Suppose $t = \phi(\tau)$ with $\alpha \leq \tau \leq \beta$, where $\phi$ is a real-valued function mapping $\alpha \leq \tau \leq \beta$ to $a \leq t \leq b$. Suppose (these are the conditions mentioned above) $\phi$ is continuous, has a continuous derivative, and $\phi'(\tau) > 0$ for each $\tau$ in its domain. Then

$$\int_a^b |z'(t)| \, dt = \int_\alpha^\beta |Z'(\tau)| \, d\tau$$

where $Z(t) := z(\phi(\tau))$.

**Problem 225.** Does the transformation $\tau = t^2$ of problem 217 satisfy the conditions in this theorem?

**Definition 226.** Let $z(t)$ be an arc on $a \leq t \leq b$ that is differentiable on $a < t < b$. If $z'(t) \neq 0$ on $a < t < b$ and $z'(t)$ is continuous on $a \leq t \leq b$, then $z$ is said to be smooth.

**Exercise 227.** Verify:

1. The function $z(t) := (\cos t, \sin t)$ is smooth. Also, plot the graph of $z(t)$. Finally, treating $z'(t)$ as a vector with its base at $z(t)$ at time $t$, plot $z'(t)$ on the same graph for the times $0, \pi/4, \pi/2$, etc.. (For example, plot the vector $z'(\pi/4)$ with its base at $z(\pi/4)$.) What do you notice geometrically about $z'(t)$ compared to $z(t)$ interpreted as vectors?

2. The function $z(t) := (t^2, t^3)$ on $-1 \leq t \leq 1$ is not smooth.

There are also functions whose derivatives are not continuous at every point in their domain.

The following definition will cover all cases of interest to us.
Definition 228. A contour on $a \leq t \leq b$ is a piecewise function consisting of a finite number of smooth arcs $z_k(t)$ on $a_k \leq t \leq b_k$, $1 \leq k \leq n$ such that $z_{k-1}(b_{k-1}) = z_k(a_k)$, $2 \leq k \leq n$. Contours are also called piecewise smooth arcs and are often denoted by $C$ or $z(t)$. If the contour $C$ has the same values at $t = a$ and $t = b$ and for no other values of $t$, then the contour is a simple closed contour.

Informally, contours are arcs joined end to end.

Exercise 229. Which of the following are contours? simple closed contours?

1. $z(t) := \begin{cases} 2 + it & \text{if } 0 \leq t \leq 2 \\ t + 2i & \text{if } 2 \leq t \leq 3 \end{cases}$ on $0 \leq t \leq 3$.

2. $z(\theta) := (\cos \theta, \sin \theta)$ on $0 \leq \theta \leq 2\pi$.

3. $z(\theta) := (\cos \theta, \sin \theta)$ on $0 \leq \theta \leq 4\pi$.

Exercise 230. The following contours $C_1$, $C_2$, and $C_3$ are given graphically.

Note that the direction of the arrows indicates the direction of the mapping defining the arc and it is assumed the contours are one-to-one except at the end points.

1. Describe each contour algebraically.

2. Let $C$ be given by the closed clockwise contour around the entire graph in the last exercise. Write $C$ algebraically. (Hint: piecewise function.)

Definition 231. Suppose $C_1(t)$, $a_1 \leq t \leq b_1$, and $C_2(t)$, $a_2 \leq t \leq b_2$, are two contours such that $C_1(b_1) = C_2(a_2)$. Then the sum of $C_1$ and $C_2$ is the contour given by the mapping

$$\begin{cases} C_1(t), & a_1 \leq t \leq b_1 \\ C_2(a_2 + (t - b_1)), & b_1 \leq t \leq b_1 + (b_2 - a_2) \end{cases}$$

with domain $a_1 \leq t \leq b_1 + (b_2 - a_2)$ and denoted by $C_1 + C_2$. 

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Exercise 232. Verify $C_1 + C_2$ in the last definition is a contour.

Theorem 233 (Chain Rule). Suppose that a function $f$ is analytic at a point $z_0 = z(t_0)$ contained in the range of a smooth arc $z(t)$ ($a \leq t \leq b$). Show that if $w(t) := f[z(t)]$, then

$$w'(t_0) = f'(z_0)z'(t_0).$$

See remark below.

Remark 234. We need to be a little careful because the notation is somewhat sloppy. Here $'$ means two things, depending on the function. $w'(t_0)$ is the derivative of $w$ with respect to $t$ at $t_0$. $f'(z_0)$ is the derivative of $f$ with respect to $z$ at $z_0$.

[Hint: Write $f(z) = u(x,y) + iv(x,y)$, where $z(t) = x(t) + iy(t)$. Apply the chain rule in multivariable calculus for functions of two variables. To help you recall, if $w(t) := f(x(t), y(t))$, then the chain rule says $w'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$ or written another way

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

at value $t$ in the domain of $w$.]

4.3 Contour Integrals

Problem 235. Review the definition of a line integral from multi-variable calculus. Describe here what it is.

Contour integrals are essentially line integrals.

Definition 236. Let $z(t)$, $a \leq t \leq b$, represent a contour $C$ from the point $z_1 := z(a)$ to the point $z_2 := z(b)$. Let $f$ be piecewise continuous on a domain containing the range of $C$. The contour integral of $f$ along $C$ is defined by

$$\int_a^b f(z(t))z'(t)dt,$$

denoted

$$\int_C f(z)dz.$$

Remark 237. If $f$ has has some property $P$ (for example continuous) on a domain containing the range of a contour $C$, we will express this by saying “$f$ is $P$ (e.g. continuous) on $C$”.

Remark 238. To understand $f(z(t))$, recall the definition of a complex number as an ordered pair. Consider $f(z)$ as $f(x,y)$, where $z$ is interpreted as an ordered pair $z := (x,y)$. Thus, $f(z(t))$ means substituting $x(t)$ in for $x$ and $y(t)$ in for $y$.

Remark 239. One way to help remember the integral is to re-write the right-hand integral as

$$\int_a^b f[z(t)]\frac{dz}{dt}dt$$

and to think of $\frac{dz}{dt} = 1$ to obtain the integral being defined.
Exercise 240. Suppose $f(z) := x + iy^2$ and a contour is given by $z(t) := e^t + it$ on $a \leq t \leq b$. Determine $x(t)$, $y(t)$, and $f(z(t))$.

Problem 241. This problem is to induce you to think about why we have the conditions on $f$ and why the contour was defined the way it was.

1. Recall from calculus that continuous functions are integrable. Look up this fact in your calculus text and explain why real continuous functions are integrable. It is a small step to argue then that real piecewise continuous functions are integrable. Explain this also.

2. In the above definition, $f$ is specified to be piecewise continuous on $C$ so that it is integrable. Also, recall that $z(t)$ and $z'(t)$ are piecewise continuous, along the contour. Why? Thus, $f[z(t)]z'(t)$ is integrable. Why?

Exercise 242. Roughly sketch the range and direction of the contour in the plane and calculate the integral

$$\int_C f(z) dz$$

for each of the following:

1. For $C$ given by $z(t) := t^3 + it$ on $0 \leq t \leq 2$ and for $f(z) := x + iy^2$, given in terms of $x$ and $y$.

2. For $C$ given by $z(t) := t + it^2$, $1 \leq t \leq 3$ and for $f(z) := z^2$.

3. For $C$ given by $z(\theta) := e^{i\theta}$, $-\pi \leq \theta \leq \pi$ and for $f(z) := z$.

4. For $C$ given by $z(\theta) := e^{i\theta}$, $-\pi \leq \theta \leq \pi$ and for $f(z) := \overline{z}$.

5. For $C$ given by $z(\theta) := e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$ and for $f(z) := z^2 + 1$.

6. For $C$ given by the simple arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$ and for

$$f(z) := \begin{cases} 
1 & \text{if } y < 0 \\
4y & \text{if } y > 0,
\end{cases}$$

Exercise 243. Calculate the contour integral for $C$ given by $3e^{i\theta}$, $\pi/2 \leq \theta \leq \pi$, and for $f(z) := z^{\frac{3}{2}}$ on the branch $0 < \theta < 2\pi$.

Theorem 244. Let $f$ be piecewise continuous on contour $C$ and $c_0 \in C$. Then

1. $\int_C c_0 f(z) dz = c_0 \int_C f(z) dz$

2. $\int_C [f(z) + g(z)] dz = \int_C f(z) dz + \int_C g(z) dz$

Problem 245. Contour $C$ versus $-C$. 

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1. Let the contour $C$ be $1 + t$ with $a = -1$ and $b = 2$. Sketch this contour. What points in the $z$-plane does this contour start and end at?

2. Let the contour $-C$ be $1 - t$ with $a = -2$ and $b = 1$. Sketch this contour. What points in the $z$-plane does this contour start and end at?

3. How is $-C$ related to $C$?

**Definition 246.** Let $z(t)$, $a \leq t \leq b$, represent a contour $C$ from the point $z_1 := z(a)$ to the point $z_2 := z(b)$. Define the contour $-C$ to be $z(-t)$, $-b \leq t \leq -a$.

**Theorem 247.** Let $z(t)$, $a \leq t \leq b$, represent a contour $C$ from the point $z_1 := z(a)$ to the point $z_2 := z(b)$. Let $f$ be piecewise continuous on $C$, meaning $f[z(t)]$ is continuous on $a \leq t \leq b$. Then

$$\int_{-C} f(z) \, dz = -\int_C f(z(t)) z'(t) \, dt.$$  

(Hint: Use problem 192).

**Problem 248.** The relationship between $\int_C$ and $\int_{-C}$ is very analogous to the relationship between two integrals on the real line. What two integrals?

The following problem is worth noting:

**Problem 249.** Let $f(z) := z$. Let $C$ be the contour

$$z(t) = \begin{cases} 
  it & \text{if } 0 \leq t \leq 2 \\
  2(t-2) + 2i & \text{if } 2 \leq t \leq 4
\end{cases}$$

Let $C_1$ be the contour $z(t) := it$ on $0 \leq t \leq 2$. Let $C_2$ be the contour $z(t) := 2(t-2) + 2i$ on $2 \leq t \leq 4$. Calculate

$$\int_C f(z) \, dz, \quad \int_{C_1} f(z) \, dz, \quad \int_{C_2} f(z) \, dz$$

What is the relationship between these integrals?

The following theorem arises from exactly the same idea as the last problem:

**Theorem 250.** Let $C_1$ be a contour from $z_1$ to $z_2$. Let $C_2$ be a contour from $z_2$ to $z_3$. Let $C$ be the contour from $z_1$ to $z_3$ along $C_1$ and then $C_2$. Let $f$ be a piecewise continuous function on $C$. Then

$$\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz$$

**Notation 251.** By the last theorem, a common way of writing

$$\int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz \quad \text{is} \quad \int_{C_1+C_2} f(z) \, dz.$$

**Problem 252.** Path dependence of contour integrals.

1. Suppose we have the contours given in exercise 230. Let $f(z) := y - x - i3x^2$. Calculate

$$\int_{-C_1} f \, dz, \quad \text{and} \quad \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz$$
2. Is \[ \int_{-C_3} f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz? \]

Observe \(-C_3\) and \(C_1 + C_2\) have the same endpoints. Thus, the integral depends on the path!

**Problem 253.** Suppose we have a contour \(C\) with endpoints \(z_1\) and \(z_2\) and the function \(f(z) = z\). Can you determine \[ \int_C z \, dz \]

in terms of the endpoints \(z_1\) and \(z_2\)? (Hint: \(z(t)z'(t) = \frac{1}{2} i \pi (z(t))^2\).)

From problem 252, we found that in general the integral depends on the path. However, this last problem shows that sometimes the integral is *independent of path*. We will determine conditions where the contour is path independent soon.

Some results that may prove useful for subsequent problems in the future:

**Theorem 254.**

1. Let \(z_0 \in \mathbb{C}\). Let \(C\) and \(C_0\) denote the circles of radius \(R\), \(Re^{i\theta} (-\pi \leq \theta \leq \pi)\) and \(z_0 + Re^{i\theta} (-\pi \leq \theta \leq \pi)\), respectively, with \(R > 0\). Then

\[ \int_C f(z) \, dz = \int_{C_0} f(z - z_0) \, dz \]

for any piecewise continuous function \(f\) on \(C\).

2. Let \(C_0\) be the same circle as in the first part of this theorem. Show

\[ \int_{C_0} (z - z_0)^{n-1} \, dz = \begin{cases} 2\pi i & \text{if } n = 0 \\ 0 & \text{if } n = \pm 1, \pm 2, \ldots \end{cases} \]

This next theorem is very important in analysis.

**Theorem 255** (Upper Bounds for Moduli of Contour Integrals). Let \(C\) be a contour from \(a\) to \(b\) and \(f\) a piecewise continuous function on \(C\). Suppose \(|f(z)| \leq M\) on \(C\) for some constant \(M \geq 0\). Then

\[ \left| \int_C f(z) \, dz \right| \leq ML \]

where \(L\) is the length of the contour.

Hint: Applying the definition of contour integration and then using a previous theorem involving the modulus may be helpful.

In the following exercises, you may need to use the triangle inequality, and its extension, namely

\[ |z_1| + |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2| \]

For example, \[ |z| + |1 + i| \leq |z + (1 + i)| \leq |z| + |1 + i| \].

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Exercise 256. Convince yourself of this extension of the triangle inequality by trying various real, both positive and negative, values for \( z_i \).

The proof of this extension follows rather easily from the triangle inequality but you are not required to prove it here.

Exercise 257. Let \( C \) be that part of the circle that is in the first and second quadrant \( 3e^{it} \), \( 0 \leq t \leq \pi \).

1. Show that
\[
\left| \int_C (z + 3) \, dz \right| \leq 18\pi
\]

2. Show that
\[
\left| \int_C \frac{z + 6}{z^2 - 1} \, dz \right| \leq \frac{27\pi}{8}
\]

Hint: Bounding the numerator by something larger and the denominator by something smaller may be useful in bounding the entire integrand by something larger.

Exercise 258. Let \( C_R \) be the circle \( Re^{i\theta} \), \( 0 \leq \theta \leq \pi \). Let \( \sqrt{z} \) denote the principal branch of \( z^{\frac{1}{2}} \). Show that
\[
\lim_{R \to 0} \left| \int_{C_R} \frac{\sqrt{z}}{z^2 + 5} \, dz \right| = 0
\]

This type of result becomes important when evaluating contour integrals.

4.4 Antiderivatives

Exercise 259. You are given \( f(z) := z \) and the following contours \( C_1, C_2, \) and \( C_3 \).
Integrals

1. Determine \( \int_{C_1+C_2} f(z)dz \)
   
   by direct contour integration and also by using problem 253.

2. Determine \( \int_{C_1+C_2+C_3} f(z)dz \)
   
   by direct contour integration and also by using problem 253.

This last problem illustrates the idea behind antiderivatives.

**Definition 260.** Let \( f \) be a continuous complex function on a domain \( D \). An antiderivative of \( f \) is a function \( F \) such that \( F' = f \) on \( D \), i.e. \( F'(z) = f(z) \), \( \forall z \in D \).

**Exercise 261.** For each \( n \in \mathbb{N} \), there is an antiderivative of \( f(z) = z^n \) over some domain. Determine such an antiderivative. Over what domain is it valid? Does its existence and domain depend on \( n \)? How?

**Theorem 262** (Fundamental Theorem of Calculus (FTC) for Functions of a Complex Variable). Conjecture what the Fundamental Theorem of Calculus for Functions of a Complex Variable should be based on the reals case or by comparing to theorem 199.

**Exercise 263.** For the first two problems, \( n = 0, 1, 2, ... \).

1. Using the FTC, determine \( \int_C z^n dz \) for any contour \( C \) connecting \(-1 + i \) to \( 3 - i \).

2. Using the FTC, determine \( \int_C z^n dz \) for any contour \( C \) connecting \( z_1 \) to \( z_2 \).

3. Define \( f(z) := 1/z^2 \), \( z \neq 0 \). There is an antiderivative of \( f(z) = 1/z^2 \) on some domain (open connected set). Determine such an antiderivative. Over what domain(s) is it valid? Now calculate \( \int_C 1/z^2 dz \) where \( C \) is the contour \( z = 2e^{i\theta} \) for \( -\pi \leq \theta \leq \pi \).

4. Let \( f(z) := 1/z \), \( z \neq 0 \). We will calculate the contour integral of this function around a circle \( 3e^{i\theta} \), \( -\pi/2 \leq \theta \leq 3\pi/2 \), in several steps using antiderivatives.

   (a) Let \( F \) be any branch of log. Then \( F \) is an antiderivative of \( f \) where the domain of \( f \) is restricted to the domain of \( F \) (by removing the branch cut from the domain of \( f \)).

   (b) By choosing the principal branch of the log as the antiderivative, apply the FTC to calculate the integral \( \int_{C_1} 1/z dz \)

   where \( C_1 \) is the half-circle \( z(\theta) := 3e^{i\theta} \), \( -\pi/2 \leq \theta \leq \pi/2 \).
(c) Suppose $C_2$ is the half-circle $z(\theta) := \exp(i\theta), (\pi/2 \leq \theta \leq 3\pi/2)$. Could you apply the FTC to calculate the integral

$$\int_{C_2} \frac{1}{z} \, dz$$

using the principal branch of the log for the antiderivative? Why not?

(d) Choosing the branch $0 < \theta < 2\pi$ of log, apply the FTC to calculate the integral

$$\int_{C_2} \frac{1}{z} \, dz.$$

(e) Let $C := C_1 + C_2$. (Observe the "path" is just a clockwise circle around the origin.) Use the last two integrals to determine

$$\int_{C} \frac{1}{z} \, dz.$$

(Hint: This integral is a sum of the two integrals you just determined!) Compare to theorem 254 part 2 with $z_0 = 0$. You should obtain the same result.

5. By considering two different branches of the antiderivative of $z^{1/2}$ and using the FTC, calculate

$$\int_{C_1 + C_2} z^{1/2} \, dz$$

where $C_1$ is any contour in the left half plane connecting $z = -i$ to $z = 2i$ and $C_2$ is any contour in the right half plane connecting $z = 2i$ to $z = -i$.

Lesson: An antiderivative may depend on the contour being integrated over.

Notice how identifying the domain of the antiderivative of the function $f$ (as determined by the branch) is a critical part of using the antiderivative to determine the integral.

**Problem 264.** What is the benefit of calculating the contour integrals with antiderivatives rather than by directly along the contour as in the previous section, even when different branches have to be used?

The next theorem generalizes the observation in the third exercise above. For the following theorem, prove the first statement implies the second, the second implies the third, and the third implies the second. We will work together in class on showing that the second implies the first.

**Theorem 265.** Suppose that a function $f(z)$ is continuous on a domain $D$. All of the following statements are equivalent.

1. $f(z)$ has an antiderivative $F(z)$ in $D$.

2. Given any two points $z_1$ and $z_2$ in $D$ and any contours $C_a$ and $C_b$ connecting $z_1$ and $z_2$ whose range is contained in $D$, then

$$\int_{C_a} f(z) \, dz = \int_{C_b} f(z) \, dz$$

3. The integrals of $f(z)$ around closed contours whose range is contained in $D$ are 0, e.g. if $C$ is a closed contour whose range is contained in $D$, then

$$\int_{C} f(z) \, dz = 0$$
4.5 Cauchy-Goursat Theorem

We found in the last section that if \( f \) has an antiderivative in a domain \( D \), then its integral around a closed contour in \( D \) is 0. The Cauchy-Goursat Theorem provides other conditions that make this latter statement true. It becomes crucial to subsequent developments.

First some preliminary topological concepts:

**Definition 266.** A connected set is such that it cannot be the union of two disjoint open sets. A bounded set means that the set is contained in some neighborhood of zero.

Informally, a connected set is in “one piece” and a bounded set does not “approach infinity”. We will assume the Jordan curve theorem without proof:

**Theorem 267** (Jordan curve theorem). Let \( C \) be a simple closed curve in \( \mathbb{C} \). Then the complement of the range of \( C \) consists of two disjoint connected sets. The bounded set is called the interior of \( C \) and the unbounded set is called the exterior of \( C \).

Because contours are functions, to be more precise we should say the interior of the range of \( C \) but we will use the term the interior of \( C \) as stated in the Jordan Curve Theorem. This is in the same spirit as remark 237.

**Exercise 268.** Draw a simple closed contour \( C \) and indicate where the interior and exterior of \( C \) are.

**Definition 269.** Suppose \( x \) and \( y \) are differentiable on \( a \leq t \leq b \). Also suppose \( u \) and \( v \) are continuous on their domains and that the range of \( (x,y) \) are contained in the domain of \( u \) and \( v \). Then, recall from multivariable calculus or advanced calculus the line integral definition

\[
\int_{a}^{b} \left( u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t) \right) dt \quad \text{denoted} \quad \int_{C} u \, dx - v \, dy.
\]

**Lemma 270.** If a function \( f \) is analytic at all points in the interior of and on the contour \( C \), then

\[
\int_{C} f(z) \, dz = \int_{C} (u \, dx - v \, dy) + i \int_{C} (v \, dx + u \, dy)
\]

where \( u, v, x \) and \( y \) have the usual meaning, namely

\[ f(z) = u(x,y) + iv(x,y), \quad z = x + iy \]

Recall Green’s theorem from multivariable calculus, which we will accept without proof:

**Theorem 271** (Green’s Theorem). Let \( C \) be a simple closed contour. Let \( R \) be the interior of \( C \) union with \( C \). Let \( P(x,y) \) and \( Q(x,y) \) be two real-valued functions on the plane that are continuous on \( R \) and whose partial derivatives are continuous on \( R \). Then

\[
\int_{C} P \, dx + Q \, dy = \int \int_{R} (Q_{x} - P_{y}) \, dA
\]

Use the previous lemma and Green’s theorem to show the following.
Lemma 272. If a function $f$ is analytic and $f'$ is continuous at all points in the interior of and on a simple closed contour $C$, then

$$\int_C f(z)\,dz = \int \int_D (-v_x - u_y)\,dA + i \int \int_D (u_x - v_y)\,dA$$

where $D$ is the interior of $C$.

Theorem 273 (Baby Cauchy-Goursat Theorem). If a function $f$ is analytic and $f'$ is continuous at all points in the interior of and on a simple closed contour $C$, then

$$\int_C f(z)\,dz = 0$$

The following theorem removes the requirement that $f'$ be continuous. It is offered without proof.

Theorem 274 (Cauchy-Goursat Theorem). If a function $f$ is analytic at all points in the interior of and on a simple closed contour $C$, then

$$\int_C f(z)\,dz = 0$$

We will assume the intuitive geometric concepts of counterclockwise and clockwise in the following definition.

Definition 275. A simple closed contour $C$ is positively oriented if the contour is counterclockwise. A simple closed contour is negatively oriented if the contour is clockwise.

Exercise 276. Let $C$ be the negatively oriented contour $2e^{-i2\pi t}$, $0 \leq t \leq 1$. For which of the following functions does the Cauchy-Goursat theorem apply so that the integral of the function along $C$ is zero?

1. $f(z) := e^z$
2. $f(z) := \frac{z^2}{z - 5}$
3. $f(z) := \frac{z^2}{z - 1}$
4. $f(z) := \log(z + 4)$
5. $f(z) := \log(z + i)$

Problem 277. Consider the first function in the last exercise. Discuss the feasibility of determining that integral using either direct contour integration or the Fundamental Theorem of Calculus. Now discuss the usefulness of the Cauchy-Goursat Theorem.

Definition 278. A domain $D$ is simply connected if the interior of every simple closed contour in $D$ is a subset of $D$. A domain $D$ that is not simply connected is multiply connected.

A simply connected domain can be thought of one as without “holes”.

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Problem 279. Are the following simply or multiply connected?

1. The disk $|z - 3| < 2$.
2. The union of two non-intersecting disks.
3. The concentric area between the disks $|z - 3| < 1$ and $|z - 3| < 2$.

Definition 280. A contour $C$ on $a \leq t \leq b$ intersects itself if there exists $t_1, t_2 \in \mathbb{R}$ such that $t_1 \neq t_2$ and $C(t_1) = C(t_2)$. In other words, $C$ is not one-to-one.

Prove the following theorem for the two cases:

- $C$ is a simple closed contour.
- $C$ is a closed contour that intersects itself a finite number of times.

For $C$ that intersects itself infinitely number of times, it is more difficult.

Theorem 281 (Cauchy Goursat Theorem - Extended 1). If a function $f$ is analytic in a simply connected domain $D$, then

$$\int_C f(z) \, dz = 0$$

for every contour $C$ whose range is contained in $D$.

Corollary 282. A function $f$ that is analytic throughout a simply connected domain $D$ must have an antiderivative everywhere in $D$.

Theorem 283 (Cauchy Goursat Theorem - Extended 2). Suppose that

1. $C$ is a simple positively oriented closed contour;
2. $C_k$ ($k=1, 2, \ldots, n$) are simple negatively oriented closed contours whose ranges are contained in the interior of $C$, that are disjoint and whose interiors have no points in common.

(See diagram below.) If a function $f$ is analytic on all of these contours and throughout the multiply-connected domains consisting of all points in the intersection of the interior of $C$ and the exterior of each $C_k$, (e.g. the region "between" $C$ and the $C_k$'s) then

$$\int_C f(z) \, dz + \sum_{k=1}^{n} \int_{C_k} f(z) \, dz = 0$$

Note that $f$ is not required to be analytic in the interior of the $C_k$.

Hint: Draw appropriately chosen paths connecting the contours and apply the Cauchy-Goursat Theorem.
Corollary 284. Let $C_1$ be a positively oriented simple closed contour and $C_2$ a positively oriented simple closed contour whose range is contained in the interior of $C_1$. Let $f$ be a complex function analytic on $C_1$ and $C_2$, and analytic in the intersection of the interior of $C_1$ and the exterior of $C_2$. Then,

$$
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz
$$

Exercise 285. Sketch the geometric picture in the last corollary.

Problem 286. Show that, if $C$ is any positively oriented simple closed contour containing the origin then

$$
\int_C \frac{1}{z} \, dz = 2\pi i
$$

Problem 287. One common requirement of many of these theorems is that $f$ has to be analytic inside contours. Trace back the development of the theorems and determine where this requirement arose from.

### 4.6 Cauchy Integral Theorem

You will prove this by a couple of lemmas below.

Theorem 288 (Cauchy Integral Theorem). Let $f$ be analytic on a positively oriented simple closed contour $C$ and on its interior. If $z_0$ is any point in the interior of $C$, then

$$
f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz
$$

First an exercise before proving.

Exercise 289. In the theorem above, let $f(z) := 1$ for all $z$, let $z_0$ be any complex number, and let $C$ be $z_0 + Re^{2\pi i t}$, $0 \leq t \leq 1$, i.e. whose range is a circle around $z_0$ of radius $R$. You have calculated this integral before. Is the Cauchy integral theorem applicable and does it give the same answer?

Lemma 290. Let $C_\rho$ be the circle $z_0 + \rho e^{2\pi i \theta}$, $0 \leq \theta \leq 1$ where $\rho > 0$ is small enough so that $C_\rho$ is in the interior of $C$ in the Cauchy Integral Theorem. (Why does $\rho$ exist?) Then

$$
\int_{C_\rho} \frac{f(z)}{z-z_0} \, dz = \int_{C} \frac{f(z) - f(z_0)}{z-z_0} \, dz
$$

You may want to review the $\epsilon - \delta$ definition of continuity and the Upper Bounds for Moduli of Contour Integrals Theorem (theorem 255) before you prove this next lemma. 31

Lemma 291. Using the continuity of $f$, show that for each $\epsilon > 0$ there is a $\delta > 0$ such that $|z-z_0| < \delta$ implies

$$
\int_{C_\rho} \frac{f(z) - f(z_0)}{z-z_0} \, dz < 2\pi \epsilon
$$

(To do this, consider choosing $\rho < \delta$.)
Integrals

Note that this last result shows that \( \int_C \frac{f(z) - f(z_0)}{z - z_0} \, dz = 0 \). Why? Now use these lemmas to prove the Cauchy Integral Theorem. It might help you to visualize the situation by drawing a diagram with the relevant contours.

**Exercise 292.** Let \( C \) be a positively oriented simple closed curve whose range is along the lines \( x = \pm 2 \) and \( y = \pm 2 \). Evaluate each of these integrals:

1. \[ \int_C \frac{z + e^z}{z - \frac{1}{2}} \, dz \]

2. \[ \int_C \frac{z^2 + 1}{z(z^2 + 9)} \, dz \]

*Note: In all of the above, be sure to write the integrand exactly like that in the Cauchy Integral Theorem.*

The next theorem is offered without proof.

**Theorem 293.** Suppose that a function \( f \) is analytic on a positively oriented simple closed contour \( C \) and on the interior of \( C \). If \( z \) is any point in the interior of \( C \), then

\[ f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} \, ds \]

and

\[ f''(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^3} \, ds \]

Note this can be remembered by formally differentiating with respect to \( z \) under the integral sign. This, however, does NOT constitute a proof. To prove the first, you could form the difference quotient of \( f(z) \), expressed using the Cauchy Integral Theorem, determine bounds on the expression, and let \( |\Delta z| \) approach 0. To prove the second, you could form the difference quotient of \( f' \), determine bounds on the expression, and let \( |\Delta z| \) approach 0.

**Exercise 294.** Let \( C \) be any positively oriented simple closed contour. Define

\[ g(w) := \int_C \frac{z^3 - e^z}{(z - w)^3} \, dz \]

Determine \( g(w) \) when \( w \) is in the interior of \( C \) and when \( w \) is not in the interior of \( C \).

The next theorem follows from the last theorem. No calculations necessary, just a logical argument.

**Theorem 295.** If a function is analytic at a point, then its derivatives of all orders exist at that point. Moreover, all of those derivatives are analytic at that point.

**Problem 296.** This last theorem shows that analytic functions are really special. Compare this last result to differentiable real functions of real variables.
Theorem 297. 32 Suppose that a function $f$ is analytic on a positively oriented simple closed contour $C$ and on the interior of $C$. If $z$ is any point in the interior of $C$, then

$$\int_{C} \frac{f(s)}{(s-z)^{n+1}} ds = \frac{2\pi i}{n!} f^{(n)}(z)$$

for $n = 0, 1, 2, \ldots$.

If time:

Theorem 298 (Morera’s Theorem). 33 Let $f$ be continuous on a domain $D$. If

$$\int_{C} f(z) dz = 0$$

for every closed contour $C$ whose range is contained in $D$, then $f$ is analytic throughout $D$. 

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Chapter 5

Series Theorems

5.1 Series

We will assume all of the results on sequences and series from calculus. The fundamental ideas are the same here.

We already defined this in a previous section, but for completeness, it is included here.

Definition 299. \( z \) is the limit of an infinite sequence \( z_1, z_2, z_3, \ldots \) if \( \forall \varepsilon > 0, \exists \) integer \( N \) such that \( n > N \) implies \( |z_n - z| < \varepsilon \)

We then say the sequence converges to \( z \) and use notation \( \lim_{n \to \infty} z_n = z \).

We will assume the following theorem.

Theorem 300. Let \( z_n := x_n + iy_n \) for \( n = 1, 2, 3, \ldots \) and \( z := x + iy \). Then \( z_n \) converges to \( z \) if and only if \( x_n \) converges to \( x \) and \( y_n \) converges to \( y \).

Exercise 301. Decide what \( \frac{1}{n-2} + i \left( -4 + \frac{1}{n} \right) \) might converge to and then prove that it does using the last theorem.

Theorem 302.

\[
\lim_{n \to \infty} z_n = z \implies \lim_{n \to \infty} |z_n| = |z|
\]

Exercise 303. Let \( z_n := -2 + i \left( \frac{1}{n} \right)^n \). Show this converges to \(-2\). Let \( r_n \) denote the moduli and \( \Theta_n \) the principal values of the arguments of the complex numbers \( z_n \). Show that \( r_n \) converges but that \( \Theta_n \) does not.

Thus, a theorem analogous to theorem 300 for polar coordinates is not true.

Definition 304. \( S \) is the limit of an infinite series, denoted \( \sum_{n=1}^{\infty} z_n \), if \( \forall \varepsilon > 0, \exists \) integer \( N \) such that \( n > N \) implies \( |S_n - S| < \varepsilon \).
where

\[ S_n := \sum_{i=1}^{n} z_i = z_1 + z_2 + z_3 + \ldots + z_n \]

We then say the series converges to \( S \). In other words, the series converges to \( S \) if the partial sums infinite sequence \( S_n \) converges to \( S \).

**Theorem 305.** Let \( z_n := x_n + iy_n \) for \( n = 1, 2, 3, \ldots \) and \( S := X + iY \). Then

\[ \sum_{n=1}^{\infty} z_n = S \]

if and only if

\[ \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y \]

The following result from real numbers is assumed true here also:

**Theorem 306.** A necessary condition for the convergence of series

\[ \sum_{i=1}^{\infty} z_i \]

is that \( \lim_{n \to \infty} z_n = 0 \).

**Definition 307.** The series

\[ \sum_{i=1}^{\infty} z_i \]

is absolutely convergent if

\[ \sum_{i=1}^{\infty} |z_i| \]

is convergent.

We will assume the following theorem, which is the same as for real numbers as is one common proof.

**Theorem 308.** If a series is absolutely convergent, then it is convergent.

**Problem 309.** The converse of the last theorem is not true. Explain why the series \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \) shows this.

**Theorem 310.** \(^*\) Let \( c \) be a complex constant and \( z_n \) and \( w_n \) infinite sequences.

1. \( \sum_{n=1}^{\infty} z_n = S \) implies \( \sum_{n=1}^{\infty} w_n = S \)

2. \( \sum_{n=1}^{\infty} z_n = S \) implies \( \sum_{n=1}^{\infty} cz_n = cS \)
3. \( \sum_{n=1}^{\infty} z_n = S \) and \( \sum_{n=1}^{\infty} w_n = T \) implies \( \sum_{n=1}^{\infty} (z_n + w_n) = S + T \)

**Problem 311.** Show that

1. For any \( z \),
   \[
   \sum_{n=0}^{n} z^n = \frac{1 - z^{n+1}}{1 - z}
   \]
   This is a partial sum geometric series.

2. For \( |z| < 1 \)
   \[
   \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}
   \]
   (For this last one, show it using an intuitive argument, and then prove it using definition (304).) This is a geometric series.

### 5.2 Taylor’s Theorem

We will prove the following theorem at the end of this section:

**Theorem 312** (Taylor’s Theorem). Suppose that a function \( f \) is analytic throughout a disk \( |z - z_0| < R_0 \), centered at \( z_0 \) and with radius \( R_0 \). Then \( f \) has the power series representation

\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)
\]

where

\[
a_n := \frac{f^{(n)}(z_0)}{n!}
\]

That is, the series \( \sum_{n=0}^{\infty} a_n (z - z_0)^n \) converges to \( f(z) \) when \( z \) lies in the stated open disk.

When \( z_0 = 0 \), the series is called a Maclaurin Series.

**Definition 313.** The disk \( \{z \in \mathbb{C} : |z - z_0| < R_0\} \) is called the Region of Convergence.

**Exercise 314.** To figure out how to determine the region of convergence of the following, read Taylor’s Theorem again. (To determine the region of convergence means to determine \( R_0 \).)

1. If the region of convergence is \( |z| < 2 \), what is \( z_0 \) and \( R_0 \)?
2. If the region of convergence is \( |z - (2i + 1)| < 3 \), what is \( z_0 \) and \( R_0 \)?

**Problem 315.** If a function \( f \) is analytic at a point \( z_0 \), then must it have a Taylor’s series about \( z_0 \)?

**Problem 316.** If a function \( f \) is entire, then what can you say about the region of convergence of the series? (See definition 146.)
Exercise 317. Determine the Maclaurin series expansion for the following functions and determine the region of convergence.

1. $e^z$
2. $z^3 e^z$ (Hint: It follows almost immediately from the last exercise.)
3. $z^3 e^{5z}$ (Hint: It follows almost immediately from either of the last two exercises.)
4. Using the expression for geometric series:
   \[
   \frac{1}{1 - z}
   \]
5. \[
   \frac{1}{1 + z}
   \]
   (Hint: You can determine it from Taylor’s theorem, or you can use the last exercise replacing $z$ by $-z$.)
6. \[
   \frac{z^2}{z^3 + 5} = \frac{z^2}{5} \left( \frac{1}{1 + (z^3/5)} \right)
   \]
7. \[
   \frac{z^2}{(3z)^3 + 5} = \frac{z^2}{5} \left( \frac{1}{1 + (27z^3/5)} \right)
   \]

Hopefully this last exercise helps you see how to build Maclaurin and Taylor expansions using known expansions.

Exercise 318. Determine the Taylor series expansion of the following about $z_0 := 1$ and determine the region of convergence.

1. $e^z$. (Hint: Apply Taylor’s theorem or use $e^z = e^{z-1}e$ and a Maclaurin series expansion from the last exercise.)
2. $\frac{1}{z}$. (Hint: $\frac{1}{z} = \frac{1}{1 - (1/z)}$)

Now for the proof of Taylor’s theorem. We will prove it first for $z_0 := 0$.

Lemma 319. Same assumption on $f$ as in Taylor’s theorem. Let $r_0 < R_0$ and $C_0$ be the circle $r_0 e^{i \theta}$, $0 \leq \theta \leq 1$. Then, for $|z| < r_0$,

\[
 f(z) = \frac{1}{2 \pi i} \int_{C_0} \frac{f(s)}{s - z} ds
\]

Lemma 320. Same assumptions as the last lemma.

\[
 \int_{C_0} \frac{f(s)}{s - z} ds = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)}{s^{n+1}} ds z^n + \int_{C_0} \frac{f(s)}{(s - z)s^N} ds z^N
\]
(Hint: Manipulate $1/(s-z)$ so that you can apply the expression for the partial sum geometric series $\sum_{n=0}^{N-1} w^n = (1-w^N)/(1-w)$ to it with $|w| < 1$. In other words, identify what $w$ would be.)

**Lemma 321.** Same assumptions as the last lemma.

$$
\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \, ds = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} (z-z_0)^n + E_n
$$

where

$$
E_n := \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)^{n+1}} \, ds \cdot z^n
$$

approaches zero as $n \to \infty$.

(Hint: Use theorem 297 for the first part and the Upper Bound on Moduli Theorem for the second part. Also use the fact that a continuous function $f$ on an arc of finite length $C_0$ has an upper bound, say $M$, on this arc.)

Finally, prove Taylor’s theorem for general $z_0$ by a change of variables.

### 5.3 Laurent Series

**Problem 322.** Suppose you would like a series expansion of

$$
e^z z^3, \quad z \neq 0
$$

about $z_0 = 0$. To do this, multiply the Maclaurin series expansion of $e^z$ by $1/z^3$. What is the region of convergence of this expansion?

This last problem shows that there are series expansions that have negative powers, but, unlike Taylor’s series, the function that is expanded is not analytic inside an entire disk. These series are called Laurent Series.

**Definition 323.** Given $z_0 \in \mathbb{C}$ and real numbers $R_1$ and $R_2$ such that $0 \leq R_1 < R_2$, an annular domain centered at $z_0$ with inner radius $R_1$ and outer radius $R_2$ is the set $R_1 < |z-z_0| < R_2$. The boundary of a domain is the set of points which are not in the domain but that are limits of points in the domain.

Recall that domains are open connected sets. Geometrically, the boundary is the "edge" of the domain.

We may prove this theorem at the end of this section. First we’ll look at some corollaries and examples.

**Theorem 324** (Laurent Series Theorem). Suppose that a function $f$ is analytic throughout an annular domain centered at $z_0$, $R_1 < |z-z_0| < R_2$, and let $C$ denote any positively oriented simple closed contour around $z_0$ and in the interior of the boundary of the annular domain. (See diagram below.) Then, at each point $z$ in the domain, $f$ has the series representation

$$
\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \quad (R_1 < |z-z_0| < R_2)
$$
where
\[ a_n := \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (n = 0, 1, 2, \ldots) \]
and
\[ b_n := \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^n} dz \quad (n = 1, 2, \ldots) \]
That is, the series \( \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{1}{b_n} \frac{1}{(z-z_0)^n} \) converges to \( f(z) \) when \( z \) lies in the annular domain.

**Problem 325.** Show that an alternative way of expressing the Laurent series is
\[ \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad (R_1 < |z-z_0| < R_2) \]
where
\[ c_n := \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (n = 0, \pm 1, \pm 2, \ldots) \]

**Corollary 326.** If \( f \) is analytic in \( |z-z_0| < R_2 \), then
1. \( b_i = 0 \) for all positive integers \( i \).
2. The Laurent series reduces to the Taylor’s series.

Note: To determine the region of convergence of a series means to determine \( z_0, R_1 \) and \( R_2 \).

**Exercise 327.** Answer the following by comparing to the Laurent series general expression.

1. If the region of convergence is \( 0 < |z| < 2 \), what is \( z_0, R_1 \), and \( R_2 \)?
2. If the region of convergence is \( |z| < 2 \), what is \( z_0, R_1 \), and \( R_2 \)?
3. If the region of convergence is \( 2 < |z-i| < 5 \), what is \( z_0, R_1 \), and \( R_2 \)? Note that if you are to write a Laurent series with this region of convergence, you need to express the series in powers of \( |z-i| \) that converges in that domain.

**Exercise 328.** Laurent series practice:

1. Determine the Laurent series expansion of \( e^{1/z} \) about \( z_0 := 0 \) by using the MacLaurin series expansion of \( e^x \). What is its region of convergence?
2. Using one of the terms of this expansion (e.g. a particular $a_n$ or $b_n$), determine
\[ \int_C e^{1/z} \, dz \]
where $C$ is any positively oriented simple closed contour around the origin.

Exercise 329. More Laurent series practice:

1. Determine the Laurent series expansion of
\[ \frac{1}{(z - 2i)^4} \]
about $z_0 := 2i$ and determine its region of convergence.

2. Using your expansion, determine
\[ \int_C \frac{1}{(z - 2i)^4} \, dz \]
where $C$ is any positively oriented simple closed contour around $2i$. [Again, use one of the terms of this expansion (e.g. a particular $a_n$ or $b_n$)]. Compare this to the result you would get from theorem 297.

Exercise 330. More Laurent series practice:

1. Determine the Laurent series expansion of $\frac{1}{z}$ about $z_0 := 0$ and determine its region of convergence.

2. Determine the Laurent series expansion of $\frac{1}{z}$ about $z_0 := 1$ by the following steps.

   We want to make this look like the geometric series. Express the function as
\[ \frac{1}{1 - (1 - z)} \]

   Now apply the geometric series from a previous section to this.

3. Determine the region of convergence. The key point here is to recall the region of convergence for the geometric series.

4. Using your expansion, determine
\[ \int_C \frac{1}{z} \, dz \]

   where $C$ is $1 + R e^{2\pi i}$, $0 \leq t \leq 1$ and $0 < R < 1$. Why can we not let $R = 1$?

   Note how useful the geometric series was in doing this last exercise. This will often be the case.

Exercise 331. More Laurent series practice:

1. Determine the Laurent series expansion of
\[ \frac{1}{z - i} \]
about $z_0 := 0$ by the following steps: Express the function as
\[ -\frac{1}{7} \left( \frac{1}{1 - (z/i)} \right) \]

   Now apply the geometric series from the last section to this.
2. Determine its region of convergence.

3. Using your expansion, determine

\[ \int_C \frac{1}{z-i} \, dz \]

where \( C \) is Re^{2\pi t}, \( 0 \leq t \leq 1 \) and \( 0 < R < 1 \). Why can we not let \( R = 1 \)?

**Exercise 332. More Laurent series practice:**

1. Determine the Laurent series expansion of

\[ \frac{1}{z-1} \]

with region of convergence \( |z| < 1 \).

2. Determine the Laurent series expansion of

\[ \frac{1}{z-1} \]

with region of convergence \( |z| > 1 \) by expressing it in the following way:

\[ \frac{1}{z} \left( \frac{1}{1-1/z} \right) \]

Now apply the geometric series formula. Explain why this gives a series with the requested region of convergence.

**Exercise 333.** Let \( D_1 := \{ z : |z| < 1 \} \), \( D_2 := \{ z : 1 < |z| < 3 \} \), and \( D_3 := \{ z : 3 < |z| < \infty \} \).

Let

\[ f(z) := \frac{1}{(z-1)(z-3)}, \quad z \neq 1, \quad z \neq 3 \]

The key in all of the following is manipulating the rational function so that you can apply geometric series.

1. Determine the Laurent series expansion of \( f \) in \( D_1 \) by the following steps:

   (a) Form the partial fraction expansion of \( f \).

   (b) You will have a term of the form \( 1/(1-z) \) and a term of the form \( 1/(z-3) \). Just as you have done for some of the previous exercises, expand each of these in a way that is valid in \( D_1 \).

2. Determine the Laurent series expansion of \( f \) in \( D_2 \) by the following steps:

   (a) Form the (same) partial fraction expansion of \( f \).

   (b) You will have a term of the form \( 1/(1-z) \) and a term of the form \( 1/(z-3) \). Just as you have done for some of the previous exercises, expand each of these in a way that is valid in \( D_2 \).

3. Determine the Laurent series expansion of \( f \) in \( D_3 \) in a similar way.

**Exercise 334.** Determine a Laurent series expansion of

\[ \frac{1}{1+z} \]

valid in \( 1 < |z| < \infty \).
Chapter 6

Residues and Poles

6.1 Residues

We found that the integral of an analytic function around a simple closed contour is zero. We also found that if the integrand is of the form \( f(z)/(z - z_0) \) then the integral around a simple closed contour was given by Cauchy’s Integral Formula. This section addresses the question: what if the integrand isn’t analytic or in that form?

Recall the definition of singular point from definition 146.

**Definition 335.** A set \( 0 < |z - z_0| < \varepsilon \) for some \( \varepsilon > 0 \) is a deleted neighborhood of \( z_0 \). A singular point of a function \( f \) is isolated if there is a deleted neighborhood throughout which \( f \) is analytic.

**Exercise 336.** If any, determine the singular points and the isolated points of the following functions. Also, draw a picture in the \( z \)-plane indicating which points are singular and which are isolated.

1. \( \frac{z + 5}{(z + (2 + i))^3} \)

2. \( \log(z) \)

3. We haven’t defined the sin function on \( \mathbb{C} \), but think about this example just along the real axis, e.g. \( z = x \), it coincides with the usual sin function for the reals. \( \frac{1}{\sin \left( \frac{\pi}{z} \right)} \)

**Theorem 337.** If a function \( f \) has a finite number of singular points, then all of these points are isolated points.

**Theorem 338.** If \( z_0 \) is an isolated point of a function \( f \), then \( f \) can be represented by a Laurent series around \( z_0 \).

**Definition 339.** The complex number \( b_1 \)

\[
\frac{1}{2\pi i} \int_C f(z)dz
\]
Residues and Poles

in the Laurent series is called the residue of \( f \) at the isolated singular point \( z_0 \). We may also denote the residue of \( f \) at \( z_0 \) by

\[
\text{Res}_{z=z_0} f.
\]

We have been using the following regularly in the last section, but now let’s state here what we’ve been doing directly.

**Problem 340.** Suppose function \( f \) has an isolated point at \( z_0 \) and hence is analytic in a deleted neighborhood of \( z_0 \). Let \( C \) be a positively oriented simple contour in this deleted neighborhood of \( z_0 \) where \( z_0 \) is the interior of \( C \). Express the integral of \( f \) on this contour in terms of the residue.

**Exercise 341.** By applying the idea in this last problem, determine the integral

\[
\int_C \frac{1}{z(z-3)^2} \, dz
\]

where \( C \) is \( 3 + e^{2\pi i t} \), \( 0 \leq t \leq 1 \).

**Exercise 342.** Working with residues:

1. Using the Maclaurin series for \( e^z \), determine the Laurent series of \( e^{\frac{1}{z}} \) around \( z_0 := 0 \).
2. Show that

\[
\int_C e^{\frac{1}{z}} \, dz = 0
\]

where \( C \) is ANY simple closed contour around \( z_0 \).
3. Compare the last result to the Cauchy-Goursat theorem. Is the converse of the Cauchy-
Goursat theorem true?

**Exercise 343.** Determine the residue of

\[
\frac{z}{(z-1)(z-2)}
\]

1. At \( z_0 := 1 \).
2. At \( z_0 := 2 \).

### 6.2 Cauchy’s Residue Theorem

**Theorem 344** (Cauchy’s Residue Theorem). Let \( C \) be a positively oriented simple closed contour. If a function \( f \) is analytic on \( C \) and the interior of \( C \) except for a finite number of singular points \( z_k \) \((k=1,2, \ldots, n)\) in the interior of \( C \), then

\[
\int_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)
\]
Hint: Draw the contour C and draw small negatively oriented contours $C_k$ around each isolated point. (What do we mean by small?) What theorem could you now apply?

Let’s see how we apply this theorem:

**Exercise 345.** Suppose we want to integrate

$$\int_{C} \frac{7z - 4}{z(z - 1)} \, dz$$

where $C$ is $3e^{\alpha \pi t}$, $0 \leq t \leq 1$.

1. What are the isolated singular points of the integrand in the interior of C?

2. Based on the Cauchy Residue Theorem, to determine the integral we need to determine the residues at each of the singular points in interior of the contour.

   (a) Singular point $z_0 := 0$: Using the techniques we’ve been developing for Laurent Series, show that for $0 < |z| < 1$

   $$\frac{7z - 4}{z(z - 1)} = \left(7 - \frac{4}{z}\right)\left(-1 - z - z^2 - \cdots\right)$$

   What is the residue of the function at $z_0$?

   (b) Singular point $z_0 := 1$: Using the techniques we’ve been developing for Laurent Series, show that for $0 < |z - 1| < 1$

   $$\frac{7z - 4}{z(z - 1)} = \left(7 + \frac{3}{z - 1}\right)\left(1 - (z - 1) + (z - 1)^2 - \cdots\right)$$

   What is the residue of the function at $z_0$?

   Now determine the integral.

3. Alternatively, expand

   $$\frac{7z - 4}{z(z - 1)}$$

   into partial fractions and determine the separate integrals.

**Exercise 346.** Determine the following integrals, where the contour C in all cases is $5e^{\alpha \pi t}$, $0 \leq t \leq 1$.

1.

$$\int_{C} \frac{e^{-z}}{z^3} \, dz$$

2.

$$\int_{C} \frac{z + 5}{z^2 - 3z} \, dz$$

There are a variety of methods of determining residues. We’ll discuss other methods in the next section.
6.3 Residues at Poles

**Definition 347.** From the Laurent Series Theorem, we know that at an isolated singular point \( z_0 \) a function \( f \) can be represented by

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}
\]

in the punctured disk \( 0 < |z - z_0| < R \). We call the sum

\[
\sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}
\]

the Principal Part of \( f \) at \( z_0 \).

**Definition 348.** We categorize isolated singularities of a function \( f \) into three types:

1. If the principal part has a finite number of terms, say,

\[
\sum_{n=1}^{m} b_n \frac{1}{(z - z_0)^n}
\]

then \( z_0 \) is a pole of order \( m \).

2. If the principal part has no terms, then \( z_0 \) is a removable singular point.

3. If the principal part has an infinite number of terms, then \( z_0 \) is an essential singularity.

**Exercise 349.** Determine the Laurent series expansion of the following functions and classify their isolated singularities.

1. \( \frac{1 - \cos z}{z^2} \)

Assume \( \cos z \) has the same series expansion about \( 0 \) as you learned in calculus for the reals. (It does.)

2. \( \frac{z^2 + 5z - 6}{z + 5} \)

(Hint: Actually calculate by dividing \( z + 5 \) into the numerator.)

3. \( e^{1/z} \)

The following development creates a method for determining residues at poles without actually determining the Laurent expansion. We’ll look at the proof at the end.
Theorem 350. An isolated singular point \( z_0 \) of a function \( f \) is a pole of order \( m \) if and only if the mapping of \( f \) can be written in the form

\[
f(z) = \frac{\phi(z)}{(z - z_0)^m}
\]

where \( \phi(z) \) is analytic and non-zero at \( z_0 \). Moreover,

\[
\text{Res}_{z=z_0} f(z) = \phi(z_0), \quad \text{if } m = 1
\]

and

\[
\text{Res}_{z=z_0} f(z) = \frac{\phi^{m-1}(z_0)}{(m-1)!}, \quad \text{if } m \geq 2
\]

The following exercise illustrates the pertinent ideas.

Exercise 351. Determine the order of the pole, as well as \( \phi(z) \) and the residue.

1. \( z_0 := -1 \)

\[
\frac{3z^2 + 7}{z + 1}
\]

2. \( z_0 := -1/2 \)

\[
\left(\frac{z}{4z + 2}\right)^4
\]

3. \( z_0 := i \)

\[
\frac{(\log z)^5}{z^2 + 1}
\]

where the branch cut is along the positive real axis.

Exercise 352. Determine the value of the integral

\[
\int_C \frac{z^4 + 1}{(z-2)(z^2 + 16)}
\]

where \( C \) is

1. \( 1 + 2e^{2\pi i}, 0 \leq t \leq 1. \)

2. \( 7e^{2\pi i}, 0 \leq t \leq 1. \)

Hints on a proof:

1. First assume

\[
f(z) = \frac{\phi(z)}{(z - z_0)^m}
\]

where \( \phi \) is analytic at \( z = z_0 \). Write \( \phi(z) \) in a Taylor’s series expansion in some neighborhood of \( z_0 \). (Why does this series expansion exist?) Then the results follow directly by looking at this expansion.

2. To prove the converse, express \( f \) in a general Laurent expansion about \( z_0 \) assuming \( z_0 \) is a pole of order \( m \). By knowing our desired form of \( \phi \), what would \( \phi(z) \) have to be at \( z \neq z_0 \)? Now, \( \phi(z_0) \) can be found by looking at the expansion of the \( \phi \) you just found and knowing we want it to be continuous at \( z_0 \). The result will then follow.
6.4 Applications of Residue Theory - Evaluation of Improper Integrals

Recall what an improper integral of a real function is.

**Definition 353.** An improper integral of a real function $f$ over $x \geq 0$ is

$$\int_{0}^{\infty} f(x)dx := \lim_{R \to \infty} \int_{0}^{R} f(x)dx$$

The integral is said to converge to the limit if the limit exists else it diverges and the integral does not exist. An improper integral of a real function $f$ over all $x$ is

$$\int_{-\infty}^{\infty} f(x)dx := \lim_{R_{1} \to -\infty} \int_{R_{1}}^{0} f(x)dx + \lim_{R_{2} \to \infty} \int_{0}^{R_{2}} f(x)dx$$

and is said to converge if both limits exist, else it diverges, and the integral does not exist. This integral is often defined another way, namely the Cauchy Principal Value

$$PV \int_{-\infty}^{\infty} f(x)dx := \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$

Note that the two different definitions of $\int_{-\infty}^{\infty}$ can give different results. However if the first exists, then the Cauchy Principal Value exists.

**Exercise 354.** Calculate

$$\int_{-\infty}^{\infty} x^{3}dx \quad \text{and} \quad PV \int_{-\infty}^{\infty} x^{3}dx$$

using both definitions. Do they differ?

Sometimes the following is useful:

**Problem 355.** If $f$ is a real even function continuous on all $x$ that is integrable from $0$ to $\infty$, then

$$PV \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 2 \int_{0}^{\infty} f(x)dx$$

The following exercise is the kind of integral that would arise using Fourier analysis, a common technique for solving problems in applied mathematics.

**Exercise 356.** Applied math type of integral:

1. Show that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^{2} + 1)^{2}}dx = \frac{2\pi}{e^{3}}$$

by the following steps:

(a) Justify writing the last integral as

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\cos 3x}{(x^{2} + 1)^{2}}dx = \frac{2\pi}{e^{3}}$$
(b) Define
\[ f(z) := \frac{e^{iz}}{(z^2 + 1)^2} \]
and define the closed contour, \( \gamma_R = C_R + \beta_R \), where \( C_R \) consists of the upper half of the positively oriented simple closed contour whose range is the circle \( r = R \) centered at \( z = 0 \), and \( \beta_R \) is the contour along the real axis connecting \( -R \) to \( R \). (Draw a picture of these and express them algebraically.)

(c) Determine the relationship between the integrals of \( f \) along these contours.

(d) Choose \( R \) large enough to enclose any singularities of \( f \) in the upper half plane. Determine the integral of \( f \) on the closed contour \( \gamma_R \). (What would you apply to do this?)

(e) Using techniques you’ve used in the past, bound the integral of \( f \) along \( C_R \) where the bound is a function of \( R \). What happens to this bound as \( R \to \infty \)? Then what is true for
\[ \lim_{R \to \infty} \int_{\beta_R} f(z) dz \]?

(f) Deduce the original integral.

2. Determine
\[ \int_0^\infty \frac{x \sin x^2}{x^2 + 1} \, dx \]

This just touches the tip of the iceberg on evaluating integrals. There are also a number of theorems that give conditions on classes of functions ensuring that the limits of the “outer” contours approach zero. These techniques can be used to determine inverse Laplace and inverse Fourier transforms.
Notes to the Instructor

1. Together in class before the students look ahead.

2. Together in class.

3. Together in class.

4. Together in class.

5. Together in class.

6. Together in class.

7. A brief excursion into limits of sequences are included here in order to use them to show non-existence of limits.

8. It may be appropriate to spend a few minutes discussing what the definition means geometrically similar to problem 105. See the next problem also.


10. Together in class.

11. If your students are familiar with advanced calculus, then you could relate results in this section to total derivatives, directional derivatives, and Jacobians in advanced calculus.

12. Together in class before the students look ahead.

13. Together in class. Before looking ahead, have them do this and the next problem.

14. Together in class.

15. If you are short of time, consider skipping the proof of this theorem since it is very straightforward. (On the other hand, sometimes the students need a straightforward proof for confidence.)

16. Topics in this section could be related to advanced calculus.

17. Together in class.

18. The term "same path" isn't needed for the course so it isn't defined explicitly. You could remove any references to "same path" if you wish to skip that discussion.

19. Discuss these conditions together in class.

20. This exercise will also be used in problem 252. In the drawing, define $C_1$ starting at some $z_1$ ending at $z_2$, $C_2$ starting at $z_2$ ending at some $z_3$. Define $C_3$ to be any contour connecting $z_3$ to $z_1$. Make them relatively simple (e.g. line segments).

21. For this theorem, the students may need a little background on chain rule with multivariable functions. See the hint after the theorem. If you don’t want to take a detour into multivariable calculus, you could have them assume this theorem.

22. For this exercise, create the following contours so that $C_1 + C_2 + C_3$ is a simple closed
contour.

23 You will of course want to be sure they got it right.

24 You’ll probably need to guide the students in third implies first.

25 Here is another problem that may help students illustrating analogous concepts in advanced calculus: Read about Green’s theorem, conservative forces, and path independence in a section in your calculus book. Compare to this theorem. Also, explain the meaning and significance of this theorem.

26 You may want to have student’s try some examples calculating such line integrals in class if they haven’t seen them in awhile.

27 One proof is long and would need to be broken down a lot. One possibility is for you to provide it and have them read it. Particular students could explain a part of it on the board. Or skip the proof and take it as an axiom.

28 To properly define orientation for contours would require some concepts from advanced calculus, such as determinants of Jacobians, or even concepts from manifolds. We will be content with an intuitive geometric definition here.

29 To prove, after introducing polygonal paths between the $C_k$, apply Cauchy-Goursat Theorem to the two simple closed contours formed by them.

30 Draw $C$ and $C_k$ as described in the theorem, say for $n = 3$.

31 If you don’t think your students are familiar enough with strict proofs from analysis, you may need to allow less rigor in their proof of the next lemma or take it as an axiom.

32 Just to save time, you may want to skip the proof of this, but it can be proved easily by induction on $n$ using theorem 293.

33 Follows easily from Theorems 265 and 295.

34 You could make this section very short if you think your students understand series well from calculus. Again, if your students are not comfortable enough with rigorous analysis proofs, then you may want them to skip the epsilon-delta arguments and take the theorems requiring such arguments as axioms.

35 You may want to assume this theorem, which is straightforward, to save time.

36 Draw the situation described in the theorem.