Further results for the Joint Distribution of the Surplus immediately before and after Ruin under Force of Interest

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Abstract This paper studies the joint distribution of the surplus immediately before ruin and the deficit at ruin under constant force of interest. A Laplace transformation technique has been used to establish an explicit expression for the joint distribution function with zero initial reserve. Numerical computation using this alternative expression is quick and easy in the case of exponential, gamma and Pareto claim sizes. Moreover, a numerical method has been developed to efficiently approximate the joint distribution in case of non-zero initial reserve.

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1 Introduction

The ruin problem for a compound Poisson risk model with constant force of interest has been considered by several authors. Gerber, Goovaerts, and Kaas (1987) considered the probability that ruin occurs with initial surplus u. In their paper, an integral equation satisfied by the distribution of the severity of ruin was obtained. Later, Dufresne and Gerber (1988) introduced the distribution of the surplus immediately prior to ruin in the classical compound Poisson risk model. Gerber and Shiu (1997,1998) investigated the joint distribution of the time of ruin, and the deficit at ruin. Cai and Dickson (2002) studied the expected value of a discounted penalty function at ruin where the "penalty" is defined as simply a function of the surplus immediately prior to ruin and the deficit at ruin. Recently, J. Šiaulys and R. Asanavičiūtė (2006) obtained the asymptotes of the Gerber-Shiu discounted penalty function in the classical Lundberg model. Yang and Zhang (2001) studied the joint distribution of the surplus immediately before and after ruin.

In this study, using a similar method to that in Yang and Zhang (2001), an explicit expressions for the joint distribution function with zero initial reserve and constant force of interest has been found. These formulas can easily be used for computation at least in the case in which the claim distribution is exponential, gamma or Pareto. In those claim distributions the solutions of the joint distribution have been found for different parameters. Moreover, using Euler's method the joint distribution of the surplus immediately before and after ruin with non-zero initial surplus can be effectively approximated. An example illustrates the computational method for a Pareto claim size.

2 Definitions and notation

Suppose N_t is the number of claims occurring in an insurance portfolio in the time interval $(0, t]$. Assume that $\{N_t, t \geq 0\}$ is a homogeneous Poisson process with intensity λ. Let $Y_i, i = 1, 2...$ denote the claim sizes which are identically independently distributed positive random variables with common distribution function $F(y)$, where F satisfies $F(0) = 0$. Suppose the surplus at any time t is denoted by U_t. Also the claim amounts are independent of the claim number process. Let S_t denote the accumulated amount of the claims occurring in the time interval (0,t], that is, $S_t = \sum_{j=1}^{N_t} Y_j$ with $S_t = 0$ if $N_t = 0$. Assume the company has an initial reserve of u and receives premium income at rate c per unit time. Assume it also receives interest on its reserves with the force of interest δ, where δ is defined as $\delta = ln(1 + i)$ with i being the effective rate of interest. Since the surplus at future time is unknown,$\{U_t\}$ is a continuous-time stochastic process. From the preceding definitions,

$$U_t = ue^{\delta t} + c \int_0^t e^{\delta(t-v)} dv - \int_0^t e^{\delta(t-v)} dS_v.$$

The concern here is the event that *ruin* occurs, i.e., U_t becomes negative for some t. Thus the ruin time is defined as

$$T = \begin{cases} \inf\{t; U_t < 0\} & \text{if } U_t < 0 \text{ for some } t > 0 \\ \infty & \text{if } U_t \geq 0 \text{ for all } t > 0\ . \end{cases}$$

Two important random variables in connection with the time of ruin T are

U_{T+} , the deficit at ruin, and U_{T-}, the surplus immediately before the time of ruin, where $T-$ denotes the time immediately prior to ruin and $T+$ denotes the time immediately after ruin. Consider the probability

$$H_\delta(u; x, y) \quad = \quad Pr(T < \infty, U_{T+} \geq -y \text{ and } U_{T-} \leq x).$$

This is the joint distribution of the surplus immediately before ruin and the deficit at ruin under force of interest δ, where x and y are positive variables. For simplicity $H(u)$ will be used for $H_\delta(u; x, y)$. Note that with initial surplus u the ruin probability is defined as $\psi(u) = Pr(T < \infty)$. Thus, by definition, $H(u) \to \psi(u)$ as x and $y \to \infty$.

3 $H(u)$ for zero initial reserve

Cai and Dickson (2002) and Yang and Zhang (2001) studied $H(0)$, the initial value of $H(u)$ when the initial surplus is zero. The following theorem extends and complements their work, yielding an alternative expression for $H(0)$ which is numerically straightforward and easy to compute. In the following, the Laplace transform of $G(u)$ will be denoted by $\widehat{G}(s) = \displaystyle\int_0^\infty G(u)e^{-su}\, du$.
The ideas developed lead to the proof of the following central theorem.

Theorem 1.

$$H(0) = \frac{\lambda \displaystyle\int_0^\infty \widehat{C}(s) I(s)\ ds}{c \displaystyle\int_0^\infty I(s)\ ds}$$

where

$$I(s) = exp\left\{ \int_1^s \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta}\ dt \right\}$$

and

$$C(u) = [F(u+y) - F(u)]\, 1_{[0,x]}(u).$$

Proof: Ruin can not occur before the first claim with initial surplus $u \geq 0$. Therefore, consider the conditional expectation of ruin given the first claim time T_1 and the first claim amount Y_1. Given that $T_1 = t$ and $Y_1 = z$, the reserve before the first claim is $ue^{\delta t} + c\frac{e^{\delta t}-1}{\delta}$, and the reserve immediately after the first claim is $ue^{\delta t} + c\frac{e^{\delta t}-1}{\delta} - z$. From Yang and Zhang (2001), $H(u)$ can be written as

$$\begin{aligned} H(u) &= E\left[1_{[U_{T-} \leq x \text{ and } U_{T+} \geq -y, T<\infty]}\right] \\ &= E\left[E\left[1_{[U_{T-} \leq x \text{ and } U_{T+} \geq -y, T<\infty]} \mid T_1, Y_1\right]\right] \\ &= E\left[1_{[ue^{\delta T_1} + c\frac{e^{\delta T_1}-1}{\delta} - Y_1 < 0]} 1_{[U_{T_1-} \leq x \text{ and } U_{T_1+} \geq -y, T<\infty]}\right] \\ &\quad + E\left[H\left(ue^{\delta T_1} + c\frac{e^{\delta T_1}-1}{\delta} - Y_1\right) 1_{[ue^{\delta T_1} + c\frac{e^{\delta T_1}-1}{\delta} - Y_1 \geq 0]}\right]. \end{aligned}$$

Considering two different cases namely $u \leq x$ and $u > x$, the above equation can be written as

$$
\begin{aligned}
H(u) &= \int_0^{\ln[(\delta x+c)/(\delta u+c)]/\delta} \int_{ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}}^{ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}+y} \lambda e^{-\lambda t}\ dF(z)dt 1_{[0,x]}(u) \\
&+ \int_0^{\infty} \int_0^{ue^{\delta t}+c\frac{e^{\delta t}-1}{\delta}} \lambda e^{-\lambda t} H(ue^{\delta t} + c\frac{e^{\delta t}-1}{\delta} - z)\ dF(z)dt.
\end{aligned}
$$

By substituting $s = ue^{\delta t} + c\frac{e^{\delta t}-1}{\delta}$ and after simplification the above expression can be written as

$$
\begin{aligned}
H(u) &= \int_u^x \lambda(c+\delta u)^{\lambda/\delta}(c+\delta s)^{-\lambda/\delta-1}[F(s+y)-F(s)]\ ds 1_{[0,x]}(u) \\
&+ \int_u^{\infty} \int_0^s \lambda(c+\delta u)^{\lambda/\delta}(c+\delta s)^{-\lambda/\delta-1} H(s-z)dF(z)ds.
\end{aligned}
$$

$H(u)$ is absolutely continuous with respect to u. Thus, differentiation of the above expression with respect to u gives

$$
\begin{aligned}
H'(u) &= \lambda\frac{\lambda}{\delta}(c+\delta u)^{\lambda/\delta-1}\delta \int_u^x (c+\delta s)^{-\lambda/\delta-1}[F(s+y)-F(s)]\ ds 1_{[0,x]}(u) \\
&-\lambda(c+\delta u)^{\lambda/\delta}(c+\delta u)^{-\lambda/\delta-1}[F(u+y)-F(u)]1_{[0,x]}(u) \\
&+\lambda\frac{\lambda}{\delta}(c+\delta u)^{\lambda/\delta-1}\delta \int_u^{\infty}(c+\delta s)^{-\frac{\lambda}{\delta}-1} \int_0^s H(s-z)\ dF(z)\ ds \\
&-\lambda(c+\delta u)^{\frac{\lambda}{\delta}}(c+\delta u)^{-\frac{\lambda}{\delta}-1} \int_0^u H(s-z)\ dF(z).
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
(c+\delta u)H'(u) &= \lambda H(u) - \lambda \int_0^u H(u-z)\ dF(z) \\
&\quad -\lambda[F(u+y)-F(u)]1_{[0,x]}(u) \qquad (3.1)
\end{aligned}
$$

yields

$$
cH'(u)+\delta uH'(u) = \lambda H(u)-\lambda \int_0^u H(u-z)\ dF(z)-\lambda[F(u+y)-F(u)]1_{[0,x]}(u).
$$

Multiply both sides by e^{-su} and integrate with respect to u from 0 to ∞ to obtain

$$\begin{aligned}\int_0^\infty cH'(u)e^{-su}\,du + \int_0^\infty \delta u H'(u)e^{-su}\,du &= \lambda \int_0^\infty H(u)e^{-su}\,du \\ &\quad -\lambda \int_0^\infty \int_0^u H(u-z)\,dF(z)e^{-su}\,du \\ &\quad -\lambda \int_0^\infty [F(u+y)-F(u)]1_{[0,x]}(u)e^{-su}\,du\end{aligned}$$

which is, in terms of the Laplace transform,

$$c\widehat{H'}(s) + \delta\widehat{uH'}(s) = \lambda\widehat{H}(s) - \lambda\widehat{H}(s)\widehat{(dF)}(s) - \lambda\widehat{C}(s) \tag{3.2}$$

where

$$c\widehat{H'}(s) = \int_0^\infty cH'(u)e^{-su}\,du$$

and

$$C(u) = [F(u+y)-F(u)]1_{[0,x]}(u).$$

For any function $H(u)$ the transform of the first derivative property suggests

$$\widehat{H'}(s) = s\widehat{H}(s) - H(0). \tag{3.3}$$

Moreover, by the derivative-of-transform property $\widehat{uH'}(s) = -\frac{d}{ds}\widehat{H'}(s)$.
After simplification equation (3.2) can be written as

$$\frac{d}{ds}\widehat{H'}(s) + \widehat{H'}(s)\left[\frac{\lambda - \lambda\widehat{(dF)}(s) - sc}{s\delta}\right] = \lambda\frac{H(0)}{s\delta}\widehat{(dF)}(s) - \frac{\lambda H(0)}{s\delta} + \frac{\lambda\widehat{C}(s)}{\delta}.$$

This is a linear differential equation and the following lemma proves that the integrating factor $I(s) = \ exp\left\{\int_1^s \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta}\,dt\right\}$ is integrable.

Lemma 1. *If $\widehat{dF}'(0)$ is finite, then $I(s)$ is integrable.*

Proof: Consider first $I(\infty) = \lim_{s\to\infty} I(s) = \ exp\left\{\int_1^\infty \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta}\,dt\right\}$.
Because the Laplace transform $\widehat{dF}$ is positive and decreasing to zero,

$$\lim_{t\to\infty}\frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta} = \lim_{t\to\infty}\frac{\lambda - \lambda\widehat{(dF)}(t)}{t\delta} - \frac{c}{\delta} = -\frac{c}{\delta}.$$

Therefore, $\lim_{s\to\infty} \int_1^s \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta}\, dt = -\infty$ and $I(\infty) = 0$.

Now consider $I(0) = \lim_{s\to 0} I(s) = \; exp\left\{ \int_1^0 \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta}\, dt \right\}$.

Let $g = \widehat{dF}$. Since g is differentiable and $g'(0)$ is finite,

$$\lim_{t\to 0} \frac{\lambda - \lambda\widehat{(dF)}(t) - tc}{t\delta} = \lim_{t\to 0} \frac{\lambda(1-g(t))}{t\delta} - \frac{c}{\delta} = \lim_{t\to 0} \frac{\lambda g'(t)}{\delta} - \frac{c}{\delta} = \frac{\lambda g'(0) - c}{\delta}.$$

Thus, each of $I(\infty)$ and $I(0)$ is finite, concluding the proof.

Integrating from 0 to ∞ the above equation yields

$$\begin{aligned} \int_0^\infty (\widehat{H'(u)} I(s))'\, ds &= \int_0^\infty \lambda \frac{H(0)}{s\delta} \widehat{(dF)}(s) I(s)\, ds - \int_0^\infty \frac{\lambda H(0)}{s\delta} I(s)\, ds \\ &\quad + \int_0^\infty \frac{\lambda \widehat{C}(s)}{\delta} I(s)\, ds. \end{aligned}$$

Even though the natural domain of the Laplace transform is the open interval $(0, \infty)$, $\widehat{H'(u)}$ will still be defined at $s = 0$ because $H'(u)$ is integrable. Thus, the transform will be continuous from the right at $s = 0$. Thus, from equation (3.3), $\widehat{H'}(0) = -H(0)$. Moreover, from the integrability of $I(s)$ it can be concluded that $I(s)$ decays exponentially as $s \to \infty$. Solving for $H(0)$ yields

$$\begin{aligned} H(0) &= \frac{\displaystyle\int_0^\infty \widehat{C}(s) I(s)\, ds}{\displaystyle\frac{\delta I(0)}{\lambda} - \int_0^\infty \frac{1}{s}\widehat{(dF)}(s) I(s)\, ds - \int_0^\infty \frac{1}{s} I(s)\, ds} \\ &= \frac{\displaystyle\int_0^\infty \widehat{C}(s) I(s)\, ds}{\displaystyle\frac{\delta I(0)}{\lambda} + \int_0^\infty \frac{\delta}{\lambda}(I(s))'\, ds + \int_0^\infty \frac{c}{\lambda} I(s)\, ds}. \\ &= \frac{\displaystyle\lambda \int_0^\infty \widehat{C}(s) I(s)\, ds}{\displaystyle c\int_0^\infty I(s)\, ds}. \end{aligned}$$

Concluding the proof.

4 Computing $H(0)$ for different claim size distribution

The following examples deal with the exponential, gamma and Pareto claim sizes. In all of these cases the above equation of $H(0)$ has been used so

Table 4.1: Illustration of $H(0)$ for exponential(1) claims for $\lambda = 1$ and $\delta = 0.05$

Premium Income(c)	x	y	$H(0)$
3	3	3	0.29497
5	3	3	0.17872
10	3	3	0.08987
2	1	2	0.26814
2	30	30	0.47865
2	100	100	0.47870
2	200	200	0.47870
5	100	10	0.19756
5	40	40	0.19757
5	50	50	0.19757
5	100	100	0.19757

that for specific values of c, λ, δ, x and y numerical values of $H(0)$ can be calculated. The obtained values of $H(0)$ in Table 4.1-4.3 are identical to those of $H(0)$ obtained from the definition of $H(0)$ provided by Cai and Dickson (2002). However, the expression obtained here is easier to compute.

4.1 Exponential claims

Consider the exponential claim size distribution, i.e. $f(x) = \theta e^{-\theta x}, x \geq 0$. The methodology described here will work for arbitrary θ. For simplicity, set $\theta = 1$, so that the claim size has a mean 1. Maple has been employed to calculate $H(0)$ for different parameters. Maple's numerical integration by the Gaussian quadrature method has been used in all the following examples. In Table 4.1 for several values of c, x and y the evaluation of $H(0)$ has been shown when $\lambda = 1$, $\delta = 0.05$ and the claim is exponentially distributed with mean claim size 1.
It can also be observed that $H(0)$ is decreasing for fixed values of λ, δ as the values of c increasing.That is, considering all other parameters fixed if the premium rate has been increased $H(0)$ decreases. Moreover, the higher the values of x or y the higher the values of $H(0)$. Also, it can be observed that for large values of x and y, $H(0)$ converges to a number which happens to be the ruin probability with zero initial reserve, $\psi(0)$, since $H(u) \to \psi(u)$ as x and $y \to \infty$.

4.2 Gamma claims

In the line of automobile physical damage insurance, a claim event is an incident causing damage to an insured automobile. The claim amount will not have wide variability. For this reason, Bowers, Hickman, Gerber, Jones

Table 4.2: Illustration of $H(0)$ for gamma (2,2) claims for $\lambda = 1$ and $\delta = 0.05$

Premium Income(c)	x	y	$H(0)$
3	3	3	0.38368
5	3	3	0.26290
10	3	3	0.13584
2	1	2	0.13629
2	1	3	0.20744
2	10	10	0.96974
2	30	30	0.99939
2	100	100	0.99939
5	100	10	0.71564
5	50	50	0.73328
5	100	100	0.73328

and Nesbit (1989) claim that the gamma distribution has given reasonable fit to data and has been used on occasion for the claim amount distribution. Consider the gamma claim size distribution, i.e.

$$f(x;\alpha,\theta) = \frac{x^{\alpha-1}e^{-x/\theta}}{\theta^{\alpha}\Gamma\alpha}, x \geq 0.$$

This methodology will work for any α and arbitrary θ. Table 4.2 shows several values of $H(0)$ for various values of underlying parameters, namely $\alpha = 2$, $\theta = 2$,$\lambda = 1$ and $\delta = 0.05$.

From Table 4.2 it has been observed that $H(0)$ is decreasing for fixed values of x, and y as the values of c increasing. Moreover, as in the exponential case, considering all other parameters constant if the values of x and y are increased, the values of $H(0)$ increases rapidly and converges to some point (ruin probability). However, in case of gamma claim size, $H(0)$ approaches to unity very fast for lower premium rate (e.g. $c = 2$).

4.3 Pareto claims

The Pareto distribution is very useful in modeling claim sizes in insurance, due in large part to its extremely thick tail. For example, in the line of fire insurance, the claim event is a fire in an insured structure that creates a loss. Because fires cause heavy damage, adequate probability should be assigned to the higher claim amounts. In actuarial literature, Bowers, Hickman, Gerber, Jones and Nesbit (1989) suggested Pareto distribution in such case. The Pareto distribution can be used to model any variable that has a minimum value and for which the probability density decreases geometrically towards zero. Considering the case when the claim sizes are Pareto, i.e

$$f(x;\theta,\alpha) = \begin{cases} \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} & \text{if } x > \theta > 0, \alpha > 0 \\ 0 & \text{otherwise .} \end{cases}$$

Table 4.3: Illustration of $H(0)$ for Pareto claims for $\theta = 2, \lambda = 1$ and $\delta = 0.05$

Premium Income(c)	x	y	$H(0)$ with $\alpha = 0.9$	$H(0)$ with $\alpha = 2$
5	3	3	0.23252	0.38362
10	3	3	0.12916	0.19773
2	100	100	0.96367	0.99855
2	500	500	0.99131	0.99919
2	1000	1000	0.99533	0.99922
6	5	4	0.31024	0.42596
5	4000	4000	0.99476	0.71772
5	5000	5000	0.99541	0.71771

It should be noted that in the actuarial literature, most of the studies employ not the classical Pareto distribution but the Pareto distribution of the above form. As before, Maple has been used to obtain numerical solutions of $H(0)$ for different parameters. Since the mean of Pareto does exist only for $\alpha > 1$, infinite mean claim size is of special interest. Thus both the finite mean claim size ($\alpha = 2$) and infinite mean claim size ($\alpha = 0.9$) have been considered.
In Table 4.3 the larger values of x and y have been considered to show the convergence of $H(0)$. From Table 4.3 it has also been observed that $H(0)$ is a decreasing function of c for fixed values of x, and y. For smaller premium rate (e.g. $c = 2$), $H(0)$ approaches to unity very fast both in Pareto and gamma claim sizes compared to the exponential claim sizes. This behavior could be explained by the heavier tails of gamma and Pareto than that of exponential.

5 Approximating $H(u)$

In this section Euler's method has been used to approximate the joint distribution of surplus immediately before and after ruin, $H(u)$ for $u > 0$. Euler's method has been used in actuarial mathematics by several authors. For example, Dickson, Hardy and Waters (2009) evaluated policy values by solving Thiele's differential equation numerically by the Euler's method. A general description of Euler's method can be found at Burden and Faires (2004).
Equation (1) of section 3 has been used as the basis for this numerical method of approximating $H(u)$. Using the obtained value of $H(0)$ from the previous section, it's possible to find $H'(0)$ from equation (1). Considering $u = 0$ and $x > 0$ in equation (1)

$$H'(0) = \frac{\lambda H(0)}{c} - \frac{\lambda F(y)}{c}.$$

Euler's method has been employed to approximate $H(\epsilon)$ for $\epsilon > 0$. By this method $H(\epsilon)$ can be written as the value of $H(0)$ plus the time step multiplied by the slope of the function. In other words,

$$H(\epsilon) \approx H(0) + \epsilon H'(0).$$

Likewise,

$$H(2\epsilon) \approx H(\epsilon) + \epsilon H'(\epsilon).$$

Thus, $H'(\epsilon)$ needs to be approximated from equation (3.1). Here,the integral part of the equation (3.1) has been approximated by the trapezoid rule. In other words,

$$\int_0^{\epsilon} H(\epsilon - z)\, dF(z) = \left[\frac{H(\epsilon) + H(0)}{2}\right] * [F(\epsilon) - F(0)].$$

With $H(\epsilon)$ and $H'(\epsilon)$ in hand it is just a routine coding to find $H(2\epsilon)$ and $H'(2\epsilon)$ and so on. As expected the smaller the values of ϵ the better this approximation likely to be. Matlab has been used to approximate $H(u)$.

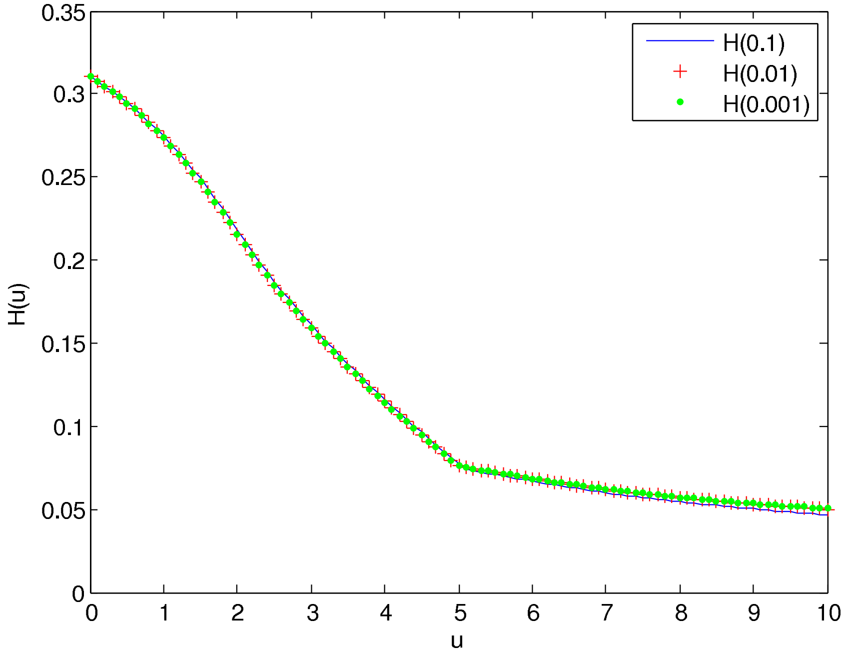
5.1 Example: Pareto claim size

The above mentioned method discussed in Section 5 can be applied to all claim size distributions. For illustration purposes Pareto claim size distribution has been considered. $H(u)$ has been numerically approximated for different step sizes, namely, $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$. For $x = 5, y = 4, c = 6, \lambda = 1, \theta = 2, \alpha = 0.9$ and $\delta = 0.05$, $H(u)$ has been approximated numerically. For the above set of parameters it has been found that $H(0) = 0.31024$ (Table 4.3). Smaller x, y values have been chosen so that the effect of the indicator function on H (Equation 3.1) can be more readily seen. However, for larger x, y values this method of approximation works even better. This numerical method depends on the fact that an effective method of computing $H(0)$ is available. Approximation of $H(u)$ has been illustrated in Figure (5.1). It has been observed that the values of $H(u)$ have settled down for different step sizes which shows that the method used to approximate $H(u)$ for $u > 0$ is effective and straightforward.

6 Concluding remarks

Cai and Dickson (2002) note that ruin functions are very complicated when $\delta > 0$. This study extends the work of Cai and Dickson (2002) and Yang and Zhang (2001) and derives the followings: an exact expression for $H(0)$ and a numerical approximation method of $H(u)$. For both occasions numerical examples have been provided. Other numerical approaches could be employed to justify the nature of $H(u)$ obtained in Section 5.

Figure 5.1: Convergence of $H(u)$ for Pareto (0.9,2) claim.



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