Seeing the Potential in Students' Questions

Rich mathematical questions emerge from online dialogues between high school algebra students and prospective teachers.

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Responding to students' questions is a critical part of teaching mathematics (Cavey, Whitenack, and Lovin 2006; NCTM 1991; Schifter 1996). A particular response may either stifle a student's inquiry or, ideally, stimulate his or her interest in mathematics. Although formulating responses that have the potential to engage students in developing new mathematical insights is challenging, we believe that this skill can be developed. We have found it helpful to view this challenge as a matter of recognizing the mathematics of a student's question—that is, the mathematics the student understands as well as the mathematics he or she may be ready to learn.

The two questions analyzed here have been used in numerous professional development seminars to engage teachers in seeing *through* the mathematical work presented by students. By making sense of a student's thought processes and anticipating the more advanced mathematical ideas that he or she might be ready to think about, we can engage both the student and the class more effectively. We describe the online dialogue from which the questions arose, analyze the two questions in the same way in which we analyze them in our professional development seminars, and conclude with guidelines for responding to students' questions that have emerged from our work with teachers.

MATHNERDS MENTORING NETWORKS

As part of its mission to support inquiry-based learning in mathematics, the nonprofit MathNerds (www.mathnerds.org) provides a free service called MathNerds Mentoring Networks (MMN), which may be used to connect a class of students to a class of prospective teachers. The students use the Web site to demonstrate the work they have done on a problem and to ask questions. Prospective teachers use the Web site to receive and respond to students' mathematical questions using an inquiry-based pedagogy by providing guidance and hints but not answers per se. These exchanges may be observed, approved, and commented on by the class teacher and the mathematics educator. During the course, prospective teachers are mentored in developing responses that allow students to resolve their own questions and consider additional mathematical ideas. The professional development activity that we describe here grew out of one of these classroom-based mentoring activities.

During professional development seminars, we frequently use sample student questions to introduce teachers to MMN. We ask the teachers, who work in small groups or individually, to formulate responses to a student's question. Teachers frequently ask whether they should develop their responses as if they were responding online or as if they were responding in person; the two situations might require different responses. In an online environment, responses should be carefully crafted to invite the student to write back with more information about how the student is thinking, whereas in a classroom setting the teacher's presence along with a simple "Tell me more" can accomplish the same objective.

Teachers then volunteer to share their responses via an overhead projector or document camera. We like to have as many groups or individuals as possible share ideas, so that many different ideas can be generated for the entire group. As ideas are shared, we conduct a group discussion about what the student seems to understand or misunderstand. Throughout the discussion, we revisit important ideas from school mathematics while attempting to understand the student's perspective.

Another important part of this discussion is to consider what mathematical insight might be associated with the student's question. That is, we want teachers to think about the mathematical ideas that the student may be ready to learn. Conversations in professional development seminars are often quite rich because of teachers' familiarity with the types of questions students typically ask and because of teachers' experiences with the conceptual challenges that students frequently face.

To simulate this experience for readers, we now present two representative exercises based on two algebra students' questions that originated in a project conducted by one of the authors.

EXAMPLE 1: SLOPE OF A VERTICAL LINE

Consider the first student's question and sample work.

Student Question
"What exactly makes the slope of a vertical line undefined?"

Student Work
"Slope of a vertical line is 1/0. Is it undefined because you can't divide anything into 0 (or no) pieces?"

In our attempt to differentiate what a student understands from what he or she does not understand, it is always worth the effort to consider carefully the exact words that the student has used. We begin with what the student understands and use that to segue into the student's question from his or her perspective.

Stop and Reflect
1. Reread the student's question and work several times.
2. What does the student understand and misunderstand about slope?

The student appears to know that the slope of a vertical line is undefined but also states that the slope of a vertical line is 1/0. With these contradictory ideas in mind, teachers are asked to think about how they might respond. The intent is to encourage teachers to consider some of the overarching mathematical ideas as well as the more advanced mathematical ideas that might be possible extensions of an ongoing dialogue with the student.

Stop and Reflect
1. What are some potential mathematical insights associated with the student's question?
2. How might you respond? Why?

Teachers who have participated in this activity have suggested responses that include the following:

- "I like your thinking! Explain your idea of slope."
- "How does slope relate to division?"
- "How did you know the slope of a vertical line is 1/0?"

Teachers have also proposed engaging the student in thinking about the slope of horizontal lines as a stepping-stone toward understanding how to work with vertical lines. Others have proposed focusing on a particular example, such as $x = 5$, and using several pairs of points on the line to compute ratios of the form

$$\frac{y_1 - y_2}{x_1 - x_2},$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

Together, these approaches create an opportunity for students to consider fractions with a numerator of zero versus fractions with a denominator of zero. When the numerator is zero, we get zero (as long as the denominator is something other than zero); however, when the denominator is zero, we get a meaningless result (vertical lines). This response might directly connect to the student's question asking whether the slope of a vertical line was "undefined because you can't divide anything into 0 (or no) pieces"

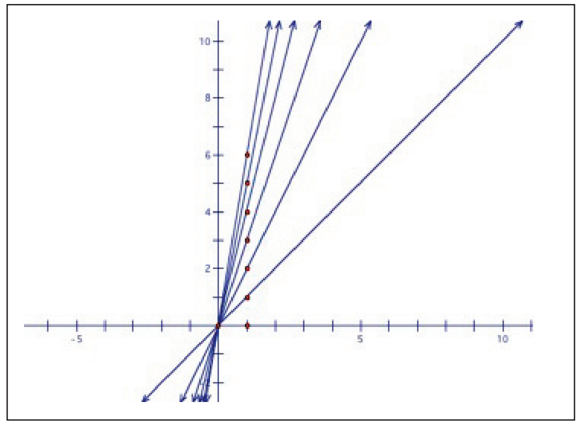

Fig. 1 A sequence of lines of the form $y = m \cdot x$. As m increases, so does the steepness of the line.

After discussing the teachers' ideas, we contribute our own to demonstrate how we might move from the student's perspective and understanding to a higher level of thinking about the mathematics. When a student computes 1/0 as a result of applying the slope formula, this is a good opportunity to point out that something must be wrong because 1/0 is undefined. If the student has applied the formula correctly, then the two points must be on the same vertical line.

Here we see an opportunity to emphasize the role of definitions in mathematics. Slope is defined only for nonvertical lines; therefore, it does not make sense to talk about slope and vertical lines in the same breath. A typical definition for slope might be, "Given a nonvertical line, the slope of the line is

$$\frac{y_1 - y_2}{x_1 - x_2},$$

where (x_1, y_1) and (x_2, y_2) are any two points on the line." Hence, the most straightforward, mathematically precise way to respond is, "Slope is undefined because slope is defined for nonvertical lines only!"

However, the essence of the student's question remains: *Why* is the definition of slope restricted to nonvertical lines? To this end, work must be done to help the student understand why it is not possible to quantitatively describe the steepness of a vertical line.

We propose engaging students in examining the slopes of a sequence of lines that get progressively steeper to develop understanding about why we do not define slope for vertical lines. A simple example of such a sequence is illustrated in the graph in **figure 1.** The first line passes through (0, 0) and (1, 1), the second through (0, 0) and (1, 2), the third through (0, 0) and (1, 3), and so on.

In light of the student's question and understanding of rise over run, it is clear that the slope of

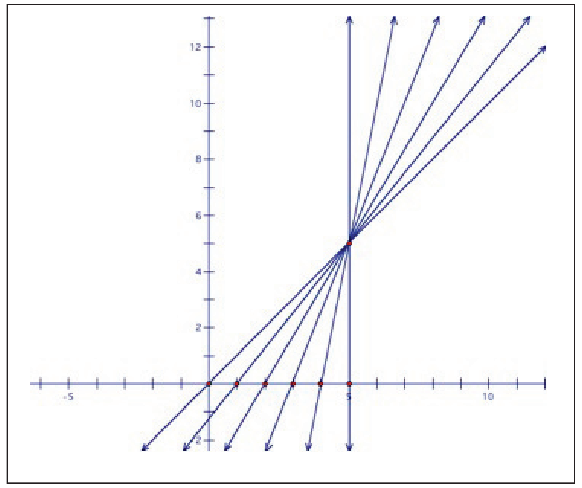

Fig. 2 As the x-intercept approaches 5, the slopes increase with bound.

Table 1 Slope Values for a Sequence of Lines Containing the Points (5, 5) and (x, 0)

x	Slope
0	1.00
1	1.25
2	1.67
3	2.5
4	5
4.5	10
4.9	50
4.95	100
4.99	500
4.995	1000
4.999	5000
4.9995	10,000
4.9999	50,000

each line is of the form

$$\frac{m - 0}{1 - 0}$$

because the points (0, 0) and (1, m) are on each line with $m = 1, 2, 3, \ldots$. So the slope value for each line is $m/1$, and as m increases, so does the slope. Notice that this sequence does not contain a vertical line. In addition, given any natural number, we can find a line in the sequence of lines of the form $y = m \cdot x$ with that slope. There is no number that can be associated with the steepness of a vertical line.

In the previous example, we purposely avoided vertical lines to make the point that there is a one-to-one correspondence between the set of real numbers and the slopes of nonvertical lines. Alternatively, one might consider a sequence of lines in which the run actually approaches zero and therefore provides a slightly different perspective on why it is not possible to define slope for vertical lines. In this case, we propose starting with the line that contains the points (0, 0) and (5, 5) and engaging students in exploring the slopes as the x-intercept moves from (0, 0) to (5, 0). This sequence of lines with integer intercepts between 0 and 5 is graphed in **figure 2**.

Using this approach, however, we have found that students have a better chance of seeing the progression after consulting a calculator or spreadsheet to compute the slope values as the x-intercept gets progressively closer to 5 (see **table 1**). With the table in hand, students can discuss how the slope values increase without bound as the x-intercept approaches 5 (from the left). We again have the situation where there are no numbers left to quantify the slope of the line $x = 5$. More generally, there is no number associated with 5/0. In either case, clearly it is not possible to define slope for vertical lines. Notice that the suggested responses give students a chance to make connections among many important concepts in mathematics, including definitions, sequences, limits, and horizontal and vertical lines.

EXAMPLE 2: CHANGING FROM SLOPE-INTERCEPT FORM TO STANDARD FORM

We briefly consider another student's question and sample work.

Student Question

"How do you change slope intercept form to standard form?"

Student Work

$$y = 6x + 12$$
$$6x = 12 + -12 = y + 12$$
$$\frac{6x}{6} = \frac{y}{6} + \frac{12}{6}$$
$$x = \frac{y}{6} + 2$$

"Is this right?"

As in the first example, the student has asked another question in the middle of the work provided—a clue that the real question may not have been carefully articulated.

Stop and Reflect

1. Reread the student's question and work provided several times.
2. What does the student understand and misunderstand?

In this case, the student clearly does not know what standard form is and wants to know whether or not he has been successful in rewriting the equation in that form. As the teachers write their responses, we ask, reemphasizing the goal of the seminars, "What does the student understand, and how can we use this to motivate the student to learn more mathematics from this question?"

Stop and Reflect

1. What are some potential mathematical insights associated with the student's question?
2. How might you respond? Why?

Teachers' responses have included the following:

- "Standard form is $ax + by = c$. How do slope-intercept form and standard form differ?"
- "Please explain what you were thinking in step 2 of your work."
- "Let's work backwards to see what you've done."

Depending on the teachers' success in discerning the nuances of the questions, we may or may not share our perspective after displaying and discussing multiple teacher responses. In this example, it is very tempting for teachers (and others) to focus on what the student has done incorrectly, therefore making it easy to overlook the impressive algebraic manipulation in his work. The student appears to be attempting to subtract 12 from both sides of the equation (despite the mistake in the second line). Because the equals and plus signs are on the same key on the keyboard, we believe that the student may have intended to write $6x + 12 + -12 = y - 12$. Granted, the student wrote $y + 12$ instead of $y - 12$, and this error might have been mathematical and not typographical. Still, the attempted idea is valid mathematics poorly executed: Get x on the left side by itself. After this, all the mathematics is correct, and if we replace $y + 12$ with $y - 12$ in the second line, then the final line would be $x = y/6 - 2$.

Although this answer is not correct, the student's work reflects good algebra and allows us to consider a more advanced mathematical idea. Note that substituting $x = 3$ into the first equation yields $y = 30$ and that substituting $y = 30$ into the corrected final equation yields $x = 3$. Hence, although the student has not placed the equation in standard form, we can praise the student for discovering what we call the *inverse function*.

We propose asking this student (as well as others) to graph $y = 6x + 12$ and $y = x/6 - 2$ and observe the graphical relationship between inverse functions. Returning to the question, we propose merely providing the definition for standard form of a line and having this student reattempt the problem. Again, note that this simple yet carefully explored mistake has resulted in opportunities for students to make connections among definitions, functions, inverse functions, and graphs of lines.

GUIDELINES FOR QUESTIONS AND RESPONSES

As a final, additional task, we ask teachers to provide general principles for responding to student questions that transcend the particular questions at hand. Without guiding the specifics of this discussion, we hope to generate a list similar to this one:

- Respond positively. Recognize some good work the student has done, such as the nice algebra in the second example, or merely comment on why the question is a particularly good one to address.
- Try to determine what the student knows. Understanding and questioning the student before directly responding to the question gives us a better chance of responding to the right question while emphasizing the importance of the question to the rest of the class.
- Connect to the student's knowledge. Once we have a better idea of what the student understands, we can use this as a springboard to connect to what the student needs to know to resolve the question at hand.
- Always provide some information. In the first example, the student needed a definition for slope of a line; in the second example, the student needed a definition for standard form for a line. With this information alone, some students would be able to resolve the question. Other students may need more guidance, but without this crucial information—even with guidance—a deep understanding of the mathematics will not likely be forthcoming.
- Try to lead the student to an understanding of the question rather than providing an answer. For example 1, having the student compute the slopes of the sequence of lines in **figure 1** is a concrete exercise that would reinforce how to compute slope and how steeper lines have larger slopes, but there is no upper bound for slopes of lines. For example 2, merely showing what standard form is and having the student reattempt the problem should be adequate.
- Connect to more advanced mathematics. In each example, we have tried to connect the student's question to mathematical concepts that will reveal themselves later in the course or in future courses. Even if the student posing the question is not ready for these next steps, there is almost certainly a student in the class who will benefit from a more advanced perspective.

FINAL COMMENTS

Our teaching experiences have led us to believe that there is no greater source for determining what a student understands and what that student may be ready to learn than a question posed by that student. The online learning community created with the mentoring networks has served as one mechanism for supporting teachers as they learn to think carefully about how to respond to students' questions. Those wishing to replicate the mentoring networks should refer to www.mathnerds.org/mathnerds/mmn. We hope that the ideas we have shared, along with the ideas you yourself generated by working through the two exercises, will encourage you to stop and carefully consider your students' questions. We have much to learn and much to gain.

Editor's note: For another take on how mathematics education students can collaborate with high school students, see the article "Mathematical Letter Writing," by Anderson Norton, Zachary Rutledge, Kareston Hall, and Rebecca Norton (*MT* December 2009/January 2010, vol. 103, no. 5, pp. 340–46); go to http//www.nctm.org/mt.

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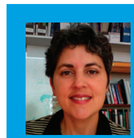

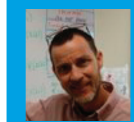

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