An alternative mathematical algorithm for the photo- and videokeratoscope

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Abstract

Due to the resolution of current laser technology, the accuracy of corneal topography as measured by the videokeratoscope is no longer adequate to provide precise enough data for refractive surgery or for the fitting of customized contact lenses. We present an algorithm for recovering corneal topography that makes use of modern differential geometric techniques and numerical descent in Sobolev spaces. We believe this algorithm may be used with the photo- and videokeratoscope to increase the accuracy of the recovered corneal topography.

1 Introduction

Accurate measurements of optical power (curvature) are required for successful refractive surgery techniques and for the fitting of customized contact lenses in order to address both higher and lower order optical abberations such as astigmatism, coma, hyperopia, myopia, and spherical abberations. The study

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of corneal topography dates to the 1880's when Plácido [1] introduced what became known as the Plácido disk. The development of the photokeratoscope and videokeratoscope advanced these techniques by merging modern computer technology with time proven techniques. See [2,3] for thorough reviews of the devices and the methods utilized by such devices for measuring corneal topography. See [4] for a discussion of the method of the alternative wave front (HS) technology for determining corneal aberrations. Turuwhenua [5] states that the videokeratoscope is still widely used in clinical practice and provides references for the commercially available devices [6-9], the principles of operation of such devices [10-12], and the clinical applications [13-16].

According to Barbosa [2], accuracy in dioptric power of .1D to .25D is to be expected. Given that accurate information concerning irregularities of the cornea is a first step toward improved contact lens manufacturing and laser surgery techniques, we offer an alternative algorithm that recovers a simulated curve to an accuracy of 10^{-5} and optical power to an accuracy of .0493 diopters.

The algorithm uses differential geometry based on the work of Oliker [17-19] to produce a differential equation in terms of the optical path length. Solving the differential equation using Sobolev steepest descent, we recover the optical path length, the curve, and the dioptric power to a high degree of accuracy. Sobolev steepest descent is a systematic preconditioning technique where gradients are based on Sobolev spaces rather than on Euclidean space, yielding superior results in terms of both time and accuracy. Introduced by Neuberger in [20], a complete discussion may be found in [21]. Problem specific applications are given in [22-27] and general references for Sobolev spaces in [28,29]. In [30] a convergence proof is given for discrete spaces such as those in this paper. For a paper concerning Sobolev gradients which are constructed based on the problem at hand, consider [31,32]. For a historical perspective on descent techniques, we direct the reader to [33] and for a general discussion consider [34,35].

2 The problem

We demonstrate our algorithm on the cylindrical target model as developed by Knoll [36], although the mathematics we present applies equally to the planar, conical, and hemispherical ring-target models described in [2]. A patient sits at the device while a cylinder of slightly larger radius than that of the human eye is placed over the eye. Illustrated in Figure 1, this cylinder has multiple rings or slits at varying heights along its periphery. Light is projected through these slits onto the cornea, then reflected through a lens at the base of the cylinder and onto a planar surface where it forms images which are circular in shape. These ring images are used to recover the shape and curvature of the cornea. We reduce the problem to two dimensions and consider the problem of recovering curves in the plane which corresponds to taking a slice of the cylinder. Place the cylinder in 3-space such that the origin is at the center of the lens and the $z-axis$ passes through the center of the cylinder. Consider the plane, illustrated

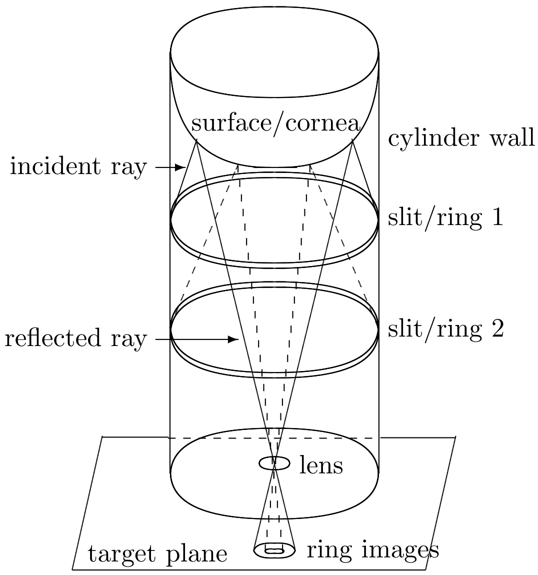

Figure 1

in Figure 2, with the $x-axis$ contained in the base of the cylinder, the origin at the center of the lens, and the $y-axis$ passing through the apex of the surface. The intersection of the cornea and the plane is now a two dimensional curve. If we may recover the curve for this plane then we may replace the plane with one which is rotated slightly, and recover the resulting curve. In this manner, we may recover the entire surface.

It is noteworthy that the mathematics applies in higher dimensions, allowing us to generate a partial differential equation and recover the full surface in one step. Returning to our planar model, for each ring we assume that we know its height and the angle which the light ray emanating from that ring makes with the $x-axis$ as it passes through the lens. This information is easily derived from the recorded ring images. From this information, we wish to recover the curve.

We make the following assumptions in 3-space.

1. The surface has a normal at every point.
2. The normal to the surface, the incident ray, and the reflected ray lie in the same plane.
3. The angle between the normal and the incident ray equals the angle between the normal and the reflected ray.
4. If p represents a point source of light on a ring of the cylinder and q represents its image on the planar surface then the function which assigns p to q is one to one.

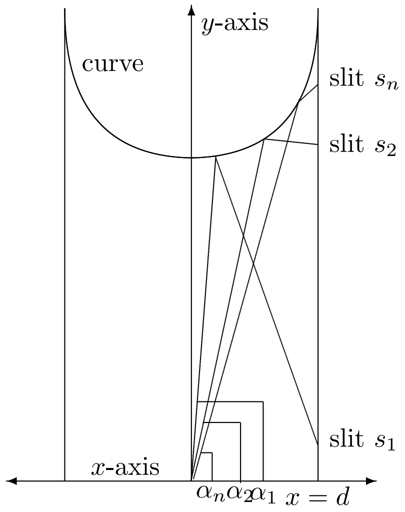

Figure 2

5. The reflected rays pass through the center of the lens.

In the plane, these assumptions may be restated.

1. The curve has a tangent at every point.

2. The angle between the incident ray and the tangent equals the angle between the reflected ray and the tangent.

3. If p is a point on the line $x = d$ and α is the angle that the reflected ray makes with the x-axis as it passes through the origin, then the function which assigns α to p is one to one.

4. The reflected ray passes through the origin.

For each ring we know its height, s, and the angle, α, the reflected ray makes with the x-axis as it passes through the lens (the origin). The method hinges on solving a differential equation to obtain the optical path length of the incident and reflected rays. The curve will be recovered from the optical path length.

3 The model

Figure 3 illustrates the two dimensional slice for the half-plane, $x > 0$. We view the rays of light as emanating not from the rings along the periphery of the cylinder, but from the center of the lens, allowing us to consider the origin of our mathematical model at the center of the lens. Let d be the radius of the cylinder and s the height of one ring of the cylinder. Thus, $\vec{R}(s) = (d, s)$ are

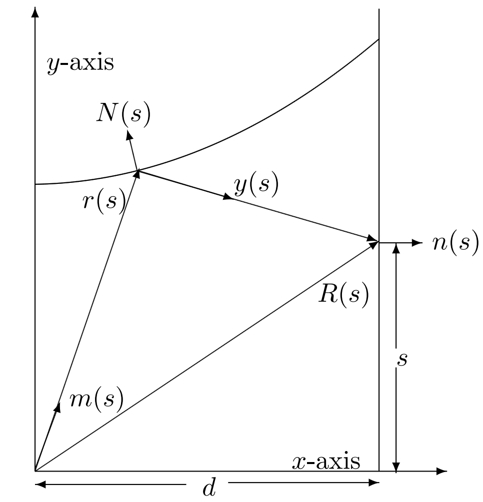

Figure 3

the coordinates of the point of the intersection of the reflected ray and the line $x = d$. Let $\vec{r}(s)$ be the point of reflection on the curve and $p(s) = \|\vec{r}(s)\|$. Let $\vec{m}(s)$ be the unit vector in the direction $\vec{r}(s)$, $\vec{y}(s)$ the unit vector in the reflected direction, and $\vec{N}(s)$ the unit normal to the curve at $\vec{r}(s)$. Let $l(s)$ be the length of the optical path. Thus, $l(s)$ is the sum of the distance from the origin to $\vec{r}(s)$ and the distance from $\vec{r}(s)$ to $\vec{R}(s)$. Let $\vec{n}(s)$ be the unit normal to the line $x = d$ at the point (d, s); $\vec{n}(s) = (1, 0)$. Two theorems provide the heart of both the derivation of the differential equation and the recovery of the curve from the resulting data from the solution of the differential equation.

Theorem 1 $\vec{y} = ((1 - \dot{l}^2)^{1/2}, \dot{l})$

Theorem 2 $p = l - \dfrac{l^2 - \vec{R}^2}{2l - 2\langle \vec{R}, \vec{y} \rangle}$

Before proving Theorems 1 and 2, we demonstrate their application to the problem. We first derive our differential equation and then recover the curve. From Figure 3,

$$\vec{R} = p\vec{m} + (l - p)\vec{y}. \tag{1}$$

Applying Theorem 2, we obtain

$$\vec{R} = p\vec{m} + \frac{(l^2 - \vec{R}^2)}{2l - 2\langle \vec{R}, \vec{y} \rangle}\vec{y}$$

and taking the dot product of both sides of this equation with $\dot{\vec{m}}$ yields,

$$\langle \vec{R}, \dot{\vec{m}} \rangle = \frac{(l^2 - \vec{R}^2)\langle \vec{y}, \dot{\vec{m}} \rangle}{2l - 2\langle \vec{R}, \vec{y} \rangle}.$$

Applying Theorem 1 and the fact that $\vec{R}(s) = (d, s)$, we see that

$$\langle \vec{R}, \dot{\vec{m}} \rangle = \frac{l^2 - s^2 - d^2}{2(l - d(1 - \dot{l}^2)^{1/2} - s\dot{l})} (\dot{m}_1 (1 - \dot{l}^2)^{1/2} + \dot{m}_2 \dot{l}) \tag{2}$$

and we have the desired differential equation in $l(s)$. It is a minor exercise to reduce Equation 2 to the standard form, $\dot{l}(s) = f(s, l, \dot{m}_1, \dot{m}_2)$. Setting (d, h) equal to the intersection of the curve with the 1ine $x = d$, we have our boundary condition: $l(h) = (d^2 + h^2)^{1/2}$.

Having derived the differential equation, assume we have a solution $l(s)$. Working in reverse order, apply first Theorem 1 to recover $\vec{y}$ then Theorem 2 to recover p. Now, $\vec{R}, p, l$, and $\vec{y}$ are known and we may rewrite Equation 1 as

$$\vec{r} = \vec{R} + p\vec{y} - l\vec{y} \tag{3}$$

to recover the curve.

Proof of Theorem 1 $\quad \vec{y} = ((1 - \dot{l}^2)^{1/2}, \dot{l})$

Claim I: $\vec{y} = \vec{m} - 2\langle \vec{m}, \vec{N} \rangle \vec{N}$

We assume from the geometry of the problem that $\vec{m}$ and $\vec{y}$ are linearly independent vectors. Thus there exist scalars α and β such that $\vec{N} = \alpha\vec{m} + \beta\vec{y}$. Hence, $\langle \vec{N}, \vec{y} \rangle = \alpha\langle \vec{m}, \vec{y} \rangle + \beta$ and $\langle \vec{N}, \vec{m} \rangle = \alpha + \beta\langle \vec{m}, \vec{y} \rangle$. From Assumption 2 Section 2, $\langle \vec{N}, \vec{y} \rangle = -\langle \vec{N}, \vec{m} \rangle$, hence $\alpha\langle \vec{m}, \vec{y} \rangle + \beta = -\alpha - \beta\langle \vec{m}, \vec{y} \rangle$ and $(\alpha + \beta)\langle \vec{m}, \vec{y} \rangle = -(\alpha + \beta)$. Either $\langle \vec{m}, \vec{y} \rangle = -1$ or $\alpha + \beta = 0$. If $\langle \vec{m}, \vec{y} \rangle = -1$, then $\langle \vec{m}, \vec{y} \rangle = |\vec{m}|\,|\vec{y}|\cos(\alpha)$ where α is the angle between $\vec{m}$ and $\vec{y}$. Thus $\langle \vec{m}, \vec{y} \rangle = -1$ implies $\alpha = \pi$ radians so $\vec{m}$ and $\vec{y}$ are not linearly independent, a contradiction. If $\alpha + \beta = 0$ then $\alpha = -\beta$ so $\vec{N} = -\beta\vec{m} + \beta\vec{y}$. If $\beta = 0$ then $\vec{N} = 0$ so the curve has no normal at this point and therefore no tangent, contradicting Assumption 1 Section 2. Assuming $\beta \neq 0$, we have $\langle \vec{N}, \vec{N} \rangle = -\beta\langle \vec{m}, \vec{N} \rangle + \beta\langle \vec{y}, \vec{N} \rangle$ or $1 = -2\beta\langle \vec{m}, \vec{N} \rangle$. This implies

$$\vec{N} = -\beta\vec{m} + \beta\vec{y} = \frac{1}{2\langle \vec{m}, \vec{N} \rangle}\vec{m} - \frac{1}{2\langle \vec{m}, \vec{N} \rangle}\vec{y}$$

and thus

$$\vec{y} = \vec{m} - 2\langle \vec{m}, \vec{N} \rangle \vec{N},$$

concluding the proof of Claim I.

Claim II: $\langle \frac{d}{ds}(\vec{R} - l\vec{y}), \vec{y} \rangle = 0$

From Claim I, $\vec{y} = \vec{m} - 2\langle \vec{m}, \vec{N} \rangle \vec{N}$ so $p\vec{y} = p\vec{m} - 2\langle p\vec{m}, \vec{N} \rangle \vec{N} = \vec{r} - 2\langle \vec{r}, \vec{N} \rangle \vec{N}$ and $\vec{r} - p\vec{y} = 2\langle \vec{r}, \vec{N} \rangle \vec{N}$. From Equation 3, $\vec{R} = \vec{r} + (l - p)\vec{y}$, hence $\vec{R} - l\vec{y} = \vec{r} - p\vec{y}$. Taking the partial with respect to s of both sides we see

$$\frac{d}{ds}(\vec{R} - l\vec{y}) \quad = \quad \frac{d}{ds}(\vec{r} - p\vec{y})$$

$$
\begin{aligned}
&= \frac{d}{ds}\left(2\langle \vec{r}, \vec{N}\rangle \vec{N}\right) \\
&= 2\langle \vec{r}, \vec{N}\rangle \frac{d\vec{N}}{ds} + 2\langle \frac{d\vec{r}}{ds}, \vec{N}\rangle \vec{N} + 2\langle \vec{r}, \frac{d\vec{N}}{ds}\rangle \vec{N} \\
&= 2\langle \vec{r}, \vec{N}\rangle \frac{d\vec{N}}{ds} + 2\langle p\vec{y} + 2\langle \vec{r}, \vec{N}\rangle \vec{N}, \frac{d\vec{N}}{ds}\rangle \vec{N} \\
&= 2\langle \vec{r}, \vec{N}\rangle \frac{d\vec{N}}{ds} + 2p\langle \vec{y}, \frac{d\vec{N}}{ds}\rangle \vec{N}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\langle \frac{d}{ds}(\vec{R} - l\vec{y}), \vec{y}\rangle &= \langle 2\langle \vec{r}, \vec{N}\rangle \frac{d\vec{N}}{ds} + 2p\langle \vec{y}, \frac{d\vec{N}}{ds}\rangle \vec{N}, \vec{y}\rangle \\
&= \langle 2\langle \vec{r}, \vec{N}\rangle \langle \vec{y}, \frac{d\vec{N}}{ds}\rangle + 2p\langle \vec{y}, \frac{d\vec{N}}{ds}\rangle \langle \vec{y}, \vec{N}\rangle \\
&= \langle 2\langle \vec{r}, \vec{N}\rangle \langle \vec{y}, \frac{d\vec{N}}{ds}\rangle - 2p\langle \vec{y}, \frac{d\vec{N}}{ds}\rangle \langle \vec{m}, \vec{N}\rangle \\
&= \langle 2\langle \vec{r}, \vec{N}\rangle \langle \vec{y}, \frac{d\vec{N}}{ds}\rangle - 2\langle \vec{y}, \frac{d\vec{N}}{ds}\rangle \langle \vec{r}, \vec{N}\rangle \\
&= 0
\end{aligned}
$$

Claim III: $\frac{dl}{ds} = \langle \frac{d\vec{R}}{ds}, \vec{y}\rangle$

$$
\begin{aligned}
0 &= \langle \frac{d}{ds}(\vec{R} - l\vec{y}), \vec{y}\rangle \\
&= \langle \frac{d\vec{R}}{ds} - \frac{dl}{ds}\vec{y} - \frac{d\vec{y}}{ds}l, \vec{y}\rangle \\
&= \langle \frac{d\vec{R}}{ds}, \vec{y}\rangle - \frac{dl}{ds}\langle \vec{y}, \vec{y}\rangle - l\langle \frac{d\vec{y}}{ds}, \vec{y}\rangle.
\end{aligned}
$$

Hence,

$$
\frac{dl}{ds} = \langle \frac{d\vec{R}}{ds}, \vec{y}\rangle.
$$

Claim IV: $\vec{y} = ((1 - \dot{l}^2)^{1/2}, \dot{l})$

Observe that $\vec{n} = (1, 0)$ and $\frac{d\vec{R}}{ds} = (1, 0)$ since $\vec{R} = (d, s)$. Thus $\frac{d\vec{R}}{ds}$ and $\vec{n}$ are linearly independent and we may write $\vec{y} = \alpha\frac{d\vec{R}}{ds} + \beta\vec{n}$ for some scalars, α and β. Taking the dot product of $\vec{y}$ first with $\frac{d\vec{R}}{ds}$ and then with $\vec{n}$ we have

$$
\langle \vec{n}, \frac{d\vec{R}}{ds}\rangle = \alpha + \beta\langle \vec{n}, \frac{d\vec{R}}{ds}\rangle = \alpha
$$

and

$$
\langle \vec{y}, \vec{n}\rangle = \alpha\langle \frac{d\vec{R}}{ds}, \vec{n}\rangle + \beta = \beta
$$

Now we may write

$$\begin{aligned}
\vec{y} &= \langle \vec{y}, \frac{d\vec{R}}{ds} \rangle \frac{d\vec{R}}{ds} + \langle \vec{y}, \vec{n} \rangle \vec{n} \\
&= \frac{dl}{ds}\frac{d\vec{R}}{ds} + \langle \vec{y}, \vec{n} \rangle \vec{n} \\
&= (\frac{dl}{ds}, \langle \vec{y}, \vec{n} \rangle).
\end{aligned}$$

Since $1 = \vec{y}^2 = \frac{dl^2}{ds} + \langle \vec{y}, \vec{n} \rangle^2$ we have, $\vec{y} = (\dot{l}, (1-\dot{l})^2)^{1/2}$, concluding the proof of Theorem 1.

Proof of Theorem 2 $\quad p = l - \dfrac{l^2 - \vec{R}^2}{2l - 2\langle \vec{R}, \vec{y} \rangle}$

We start with the Equation 1, $\vec{R} = p\vec{m} + (l-p)\vec{y}$, from which we proceed to obtain an expression involving $l^2 - \vec{R}^2$. Rearranging the terms we obtain,

$$p\vec{m} = \vec{R} - (l-p)\vec{y}$$

and squaring both sides yields,

$$p^2 = \vec{R}^2 - 2(l-p)\langle \vec{R}, \vec{y} \rangle + (l-p)^2.$$

Expanding,

$$p^2 = \vec{R}^2 - 2(l-p)\langle \vec{R}, \vec{y} \rangle + l^2 - 2lp - p^2.$$

Canceling p^2 and adding l^2 to both sides:

$$\begin{aligned}
l^2 - \vec{R}^2 &= 2l^2 - 2(l-p)\langle \vec{R}, \vec{y} \rangle - 2lp \\
&= 2l^2 - 2(l-p)\langle \vec{R}, \vec{y} \rangle - 2lp \\
&= 2[l^2 - (l-p)\langle \vec{R}, \vec{y} \rangle - lp] \\
&= 2[l^2 - l\langle \vec{R}, \vec{y} \rangle + pr\vec{y} - lp] \\
&= 2[l(l-p) - \langle \vec{R}, \vec{y} \rangle (l-p)] \\
&= 2[(l-p)(l - \langle \vec{R}, \vec{y} \rangle)]
\end{aligned}$$

Solving for p yields the desired result.

4 Numerical Considerations

We consider for our test curve a circle centered at $(0,1)$ with radius 8 millimeters since the average cornea is 7.8 millimeters. From Assumption 1 Section 2, there is a one-to-one correspondence between s and α. To generate the data, we first determine the range for α that will correspond to a range for x of .5 to 7.5 millimeters from the apex of the cornea. Then subdivide the range of α into 32 data points, and determine the corresponding values for s_i, and $\vec{m}(s_i)$. The known information is the differential equation, $\dot{l}(s) = f(s, l, \dot{\vec{m}})$, the heights, $\{s_i | i = 1, 2, \ldots, 32\}$, and the unit vectors, $\{\dot{\vec{m}}(s_i) | i = 1, 2, \ldots, 32\}$.

The method utilized for solving the differential equation is Sobolev steepest descent and references for a less condensed treatment are provided in the introduction. We first apply cubic splines to the 32 data points for α, s, and $\dot{\vec{m}}$ to extend the data to $n = 10,000$ data points. Let $\delta = 1/n$ and denote $x \in \Re^{n+1}$ by $x = (x_0, \ldots, x_n)$. Let $(\Re^{n+1}, \langle \cdot, \cdot \rangle)$ represent Euclidean space. The Euclidean inner product (dot product) is given by

$$u \cdot v =< u, v >= \sum_{i=1}^{n} u_i v_i.$$

Define discrete versions of the identity and derivative operators, D_0, D_1 : $\Re^{n+1} \to \Re^n$, by

$$D_0(x) = \begin{pmatrix} \frac{x_1 + x_2}{2} \\ \vdots \\ \frac{x_n + x_{n+1}}{2} \end{pmatrix} \quad \text{and} \quad D_1(x) = \begin{pmatrix} \frac{x_2 - x_1}{\delta} \\ \vdots \\ \frac{x_{n+1} - x_n}{\delta} \end{pmatrix}.$$

We may now modify the Euclidean inner product slightly to write:

$$\langle u, v \rangle_e = \langle D_0(u), D_0(v) \rangle = \sum_{k=1}^{n} \left(\frac{u_{k+1} + u_k}{2} \right) \left(\frac{v_{k+1} + v_k}{2} \right).$$

To consider Sobolev descent, we define a new inner product,

$$\langle u, v \rangle_s = \langle D_0(u), D_0(v) \rangle + \langle D_1(u), D_1(v) \rangle =$$

$$\sum_{k=1}^{n} \left(\frac{u_{k+1} + u_k}{2} \right) \left(\frac{v_{k+1} + v_k}{2} \right) + \left(\frac{u_{k+1} - u_k}{\delta} \right) \left(\frac{v_{k+1} - v_k}{\delta} \right).$$

Observe that this new inner product takes the derivative of the functions into consideration, providing a bit of intuition as to why it outperforms Euclidean descent in problems where the differential operator appears in the functional to be optimized. For a specific paper establishing the computational performance of Sobolev descent, see [26].

Define the perturbation space, $\Re^{n+1} = \{x \in \Re^{n+1} : x_n = 0\}$, which is the collection of vectors representing functions that are zero on the boundary. Let $(\Re^{n+1}, \langle \cdot, \cdot \rangle_s)$ denote our Sobolev space and let π_s denote the orthogonal projection of $\Re^{n+1}$ onto $\Re_0^{n+1}$ under the Sobolev inner product. Define the functional to be minimized by,

$$\phi(l) = \delta \sum_{i=1}^{n} \left(\frac{l_k - l_{k-1}}{\delta} - \frac{f_k + f_{k-1}}{2} \right)^2$$

where $f_k = f(s_k, l(s_k), \dot{\vec{m}}(s_k))$. Hence, a root of ϕ satisfying the boundary condition will provide a solution to the differential equation, $\dot{l}(s) = f(s, l, \dot{\vec{m}})$. Define the Sobolev gradient of ϕ at u, denoted $\nabla_s\phi(u)$, to be the unique element, in $(\Re^{n+1}, \langle\cdot,\cdot\rangle_s)$ satisfying $\phi'(u)(v) = \langle\nabla_s\phi(u), v\rangle_s$ for all v in $(\Re^{n+1}, \langle\cdot,\cdot\rangle_s)$. The existence of this element is guaranteed by the Riesz Representation Theorem. Choose an initial guess for the solution, l^0, satisfying the boundary condition. The process is to generate a sequence of points which converge to a point at which ϕ attains a relative minimum. The sequence is generated by setting $l^{k+1} = l^k - \delta\pi_s\nabla_s\phi(l^k)$ for $k = 0, 1, 2, \ldots$ where δ is the optimal step size. Since we seek a zero of ϕ, 'optimal' implies that $\phi\big(l - \delta\pi_s\nabla_s\phi(l)\big)$ is minimized. Because the root of ϕ will occur at the minimum where the norm of the gradient will be zero, we stop when the norm of the Sobolev gradient is less than 10^{-16}.

Upon generating the approximate solution via descent, we recover the curve via Equation 3. Then for each record we compute the curvature of the recovered curve and compare this to the true curvature. Reparameterizing the curve $\vec{r}$ with respect to arc-length, and applying the standard techniques of calculus allows us to rewrite the curvature of $\vec{r}$ with respect to the quantities, $\frac{dt}{ds}$, $\frac{d^2t}{ds^2}$, $\frac{d\vec{r}}{ds}$, and $\frac{d^2\vec{r}}{ds^2}$, as

$$\frac{d^2\vec{r}}{dt^2} = \frac{\frac{d^2\vec{r}}{ds^2}}{[\frac{dt}{ds}]^2} - \frac{\frac{d\vec{r}}{ds}\frac{d^2t}{ds^2}}{[\frac{dt}{ds}]^3}.$$

5 Results

All numerical results in this paper were produced using Microsoft Visual C++ on a 2GHz DELL Latitude D600 running WindowsXP and the code takes only a few seconds to generate the data, solve the problem, and compute the results.

Using this method and solving the problem over a range for $x \in [.5, .75]$ as measured in millimeters from the apex, we recover solve the differential equation as measured by divided difference error to an accuracy of 10^{-13}. We recover the optical length to an average error on the order of 10^{-9}, with a maximum optical error on the same order. We recover the curve to an average error of 1.5 x 10^{-5} with a maximum error of 4.6 x 10^{-5}. We recover the optical power (the dioptric error) to an average of .002828 with a maximum optical error of .0493.

In addition to the potential for improvement of accuracy of the devices in use, this method has one further advantage previously alluded to. The mathematics described in this paper is readily adapted to three dimensions. Doing so would require solving a partial differential equation, rather than the ordinary differential equation, however, the method of Sobolev descent is well-established for such equations.

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