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Calculus: The Importance of Precise Notation

William S. Mahavier and W. Ted Mahavier

Abstract: The careful use of notation and language in the statement of both the definitions and problems of calculus can begin the process of making students mathematically literate while allowing them to enjoy working on challenging problems and applications without the aid of numerous examples. Engaging the students to participate in the precise use of the language inculcates a philosophy of careful use of language that benefits all students regardless of major. Not only will this benefit those few students who decide to major in mathematics, but the precision of language will especially benefit both majors and non-majors outside of their mathematical studies. This pedagogical piece exhibits anecdotal evidence that such an approach provides motivation to students, better preparation for "proofs" classes, and a desire to pursue higher mathematics. Examples of calculus problems presented in the traditional fashion alongside subtly more precise presentations illustrate the process.

Keywords: Calculus, notation, differential.

1. INTRODUCTION

We confess: we love calculus. We became enamored of it as students, used it in industry, and have each chosen to teach one or more sections almost every year for a combined total of more than 60 years. We have contributed to calculus texts [1], developed a calculus technology laboratory [14], mentored others in the instruction of calculus [12, 24], published notes targeted at university students who took AP calculus in high school [13], and followed the literature on the reform of the subject [7, 8, 9, 10, 22]. Our love of the subject is contagious, for we have consistently attracted students from our calculus courses to take more mathematics, many continuing on to obtain advanced degrees.

Why is it that we enjoy teaching this subject so much when there seems to be so much discontent with the course from both students and faculty? Our interest in teaching calculus is closely related to what we hope to accomplish

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in the course. In addition to covering the standard syllabi, we hope to achieve the following:

1. Ensure that both we and the students enjoy the class.
2. Attract more majors to our discipline.
3. Train our students in mathematical thinking.
4. Train our students to use the language in a concise and correct manner.

In this article we emphasize the last two goals, as it is our experience that the fourth goal is a prerequisite to accomplishing the third goal, while the first two goals are consequences thereof.

As with any course we teach, we begin by considering the students. Many students who take calculus will not major in science or mathematics and will not use the computational aspects of calculus in the future. What good is such a course to them? For those majoring in the sciences or engineering, what portion of calculus has value, aside from the computational aspects that are rapidly being replaced by technology? We believe that the material of calculus is well suited to introduce students to the process of mathematical thinking, and that this process is valuable for all students, regardless of discipline. Moreover, we find that many students entering college have misconceptions about mathematics. Some plan to major in mathematics, but will change their plans when they find out what college mathematics is like. Others, like both authors, may come to college disliking mathematics due to their high school experiences, but will go on to obtain doctorates in mathematics as a result of their first college course.

In the September 2003 issue of the *Notices of the American Mathematical Society* there was a letter to the Editor by Mikhail I. Ostrovskii [20] and an article by Tony F. Chan [3]. Each expressed the opinion that good students lose interest in mathematics as a result of their calculus courses.

And Statistics from MAA's 2004 *CUPM Curriculum Guide* [21] continue to delineate a decline in the number of majors over the past fifteen years. We feel that mathematicians are won or lost on the battlefields of calculus and that this is an excellent course for attracting majors by introducing students to the mathematical way of thinking.

This idea is neither new nor original. We first saw it expressed in the introduction to *Fundamental Analysis* [18] that was used at the University of Chicago in the mid 1940s. The author points out that it is difficult to argue that knowledge of mathematics beyond arithmetic is needed for those who do not enter scientific fields of study.

> "Certainly it cannot be said, except in the case of students who intend later to enter scientific fields of study, that knowledge of abstract numbers and their properties, or of abstract geometric figures and

> their properties, is essential for the pursuit of everyday life. Why, then, the emphasis on algebra and geometry in public education? Educators, politicians, newspaper editorial writers, and the man in the street are generally agreed that if the study of these subjects has anything to offer to all students, it is *training in thinking.*"

In an article by Underwood Dudley in the *College Mathematics Journal* [5], we find a similar sentiment expressed.

> "It is time to stop claiming that mathematics is necessary for jobs. It is time to stop asserting that students must master algebra to be able to solve problems that arise every day, at home or at work. It is time to stop telling students that the main reason they should learn mathematics is that it has applications. We should not tell our students lies. They will find us out, sooner or later."

Both of these authors emphasize that it is the training in thinking that makes mathematics valuable to all and we agree. This is not meant to imply that we are advocating training in thinking at the expense of applications. To the contrary, we believe applications form the basis for such training and are key motivators for majors and non-majors alike. Indeed precision in the use of language may well be more important in applications than in "pure" mathematics. But how does one provide training in thinking? We have found that an effective way of doing this is by emphasizing the careful use of notation and language when we give students problems to solve on their own, with guidance if needed, but without the aid of numerous examples.

We believe that it is precisely this element of our courses that encourages students to think more and to take more advanced mathematics courses since most advanced courses are *very* precise in the use of the mathematical language. Calculus should lay the foundation for this precision. For majors, such precision is a field-specific imperative. For non-majors, precision of language is surely a valuable skill and these students will likely not take "transitions" or "proofs" courses to hone such a skill. What type of problems do we propose? While we may rely on our own notes, most of the time we use and assign problems from traditional texts [6, 25, 26, 27]. We often introduce different notation and devote much time to be sure that students understand exactly what they are being asked to do.

Our main theme is the need for attention, instruction, and practice in developing accurate and precise use of language, especially in basic definitions and notation in all courses, especially calculus. We require our students to carefully define the meaning of any symbols they introduce in their work and we convey to students who enjoy such depth that they may enjoy more advanced mathematical courses. Thus our insistence on the precise use of the language serves as a "carrot" to lure more students to take more mathematics. As an added bonus to the careful use of language, we find that students enjoy working on problems they completely understand. The alternative which we strive to avoid is described aptly by Selden and Selden [23] who write, "... mathematics is now often learned in small, isolated bits, which tend to be computational or procedural, devoid of conceptual understanding and largely useless in applications requiring much original thought."

Several authors have written of the importance of our theme. Askey [2] says "Do not lie to your students ..." and "Words are important and their meanings should not be changed without very good reasons." Knisley [10] says "Good theorems are the stuff of graduate courses. Good definitions are the stuff of introductory calculus." Krantz's article [11] speaks to the careful use of language in a lecture style calculus course. Two good mathematicians, Karl Menger [15] and H. S. Wall [28] wrote calculus books that abandoned the standard notation and used them in their courses, and each was highly successful in attracting students to mathematics.

Our goal in this article is to share our approach to some of the problems that experience tells us students enjoy working on with a deeper level of understanding. These are problems where the careful use of definitions and notation has motivated our students. Given the usual calculus syllabus, there is not time to go deeply into every topic, and some rote training may be appropriate. Still, we can at least distinguish in the class that which we have done with rigor satisfying any mathematician from that which we have accepted as standard practice or rote training. The result of looking deeply at a few important concepts is a greater depth of understanding of those concepts and a deeper understanding of the rigor that will be expected in upper-level courses. We encourage those students who enjoy the added rigor to take foundations, introductory algebra, or introductory analysis courses as early as possible. We schedule our own courses so that we have the opportunity to teach these follow-up classes, or we direct students to other faculty who are reputed to inspire and motivate the students.

We now proceed with a few examples to demonstrate how carefully we use language in calculus. We also offer, at the end of each section, tidbits from student evaluations which pleased us along with some of the methods we have used to measure our success.

2. RELATED RATES

Related rates are an important part of calculus since they reinforce the rules of differentiation and appear frequently in other disciplines. Our students appreciate our careful use of notation and tell us that it helps them to understand word problems. We find that our students' abilities to work word problems are directly related to their ability to carefully define the variables and functions from the problems. As an example, consider the following classical problem.

> *Assume we have a runner, running at a constant rate of 24 feet per second from home plate to 1^{st} base on a baseball diamond. Suppose the diamond is square with sides of length 90 feet. How fast is the distance from the runner to 2^{nd} base changing when the runner is half-way to 1^{st} base?*

As you review our approach, note the careful attention to the domain of the function, the chain rule, the functional notation, and the particular time of interest in the problem.

1. Let T denote the time it takes the runner to run from home plate to 1^{st} base. Let x and y be the functions such that for each number t satisfying $0 \leq t \leq T$, $x(t)$ and $y(t)$ are the distances in feet from the runner to 1^{st} and 2^{nd} base respectively.
2. If $0 \leq t \leq T$, then $y^2(t) = x^2(t) + 90^2$, so $2y(t)y'(t) = 2x(t)x'(t)$. Solving for y' yields $y'(t) = \dfrac{x(t)x'(t)}{y(t)}$.
3. Let t_0 be the time when $x(t_0) = 45$ and $y(t_0) = \sqrt{90^2 + 45^2} = 45\sqrt{5}$.
4. We may conclude that $y'(t_0) = \dfrac{x(t_0)x'(t_0)}{y(t_0)} = \dfrac{45(-24)}{45\sqrt{5}} = -\dfrac{24}{\sqrt{5}}$.

A note about the way the class is conducted may be appropriate at this point. Before presenting steps (2) through (4) in the solution above, we would set the stage by asking a number of questions of the students. As one example, we might question the assumption of constant velocity and have students sketch velocity functions that they believe to be realistic. Our students often discover that if x' is constant then they may solve directly for y and thus for y'. This is what mathematics educators refer to as a "teaching moment," the opportunity to encourage further explorations on the problem by considering a non-constant function for x'. Of course, we encourage such forays and students take comfort in the fact that the two methods result in the same solution. In the spring of 2005, three of our students gave presentations on just such explorations at the Texas sectional meeting of the Mathematical Association of America.

As you consider the traditional approach outlined below, note that each of the symbols x, y and $\frac{dy}{dx}$ are used to represent different things in the same problem. Each is used for both a function and a constant, that constant being the value of the function at a given time. The student must distinguish the appropriate meaning from the context.

1. Let x and y be the distance from the runner to 1^{st} and 2^{nd} base respectively.
2. Thus $y^2 = x^2 + 90^2$ and $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$.
3. Also $\dfrac{dy}{dt} = \dfrac{x\frac{dx}{dt}}{y}$, $\dfrac{dx}{dt} = -24$, $x = 45$, and $y = \sqrt{90^2 - 45^2} = 45\sqrt{5}$.
4. Conclude: $\dfrac{dy}{dt} = \dfrac{45(-24)}{45\sqrt{5}} = \dfrac{-24}{\sqrt{5}}$

After presenting the solution, one author warns the students that a common error is to let $x = 45$ *before* computing $\frac{dx}{dt}$ instead of after, since this gives that $\frac{dx}{dt} = 0$. The problem to which the author refers is a *direct result of the notation* since x sometimes denotes the function representing the distance from 1^{st} base and at other times represents the value of that function at a particular time. Our point in using this example is to demonstrate how unlikely it would be for a student to make this mistake using accurate notation.

In summary, we find that for student success, the notation used should include careful definitions of the functions involved (especially domains) and the time of interest (t_0 in our example). For interesting reading on even more distinct alternatives to the traditional approaches consider the papers of Menger [16, 17].

2.1. Course Evaluation Technique

On the first day of the semester, pass out a blank sheet of paper and ask each student to place a check if they are planning to minor, major, or double-major in mathematics; have them place an "x" otherwise. Repeat the exercise on the last day of class. We consistently find less than 10% checks on the first day and more than 25% checks on the last day where both percentages are taken as a percentage of the number of students on the first class day.

2.2. Course Evaluation Comments

In response to the phrase, "I believe the most effective part of the course was," one student wrote: "making us think independently about the work." Under the category, "strengths," one student wrote, "encourages students to think on their own."

3. THE INDEFINITE INTEGRAL AND *U*-SUBSTITUTION

As soon as our students are computing derivatives, we begin posing questions such as: *Can you find an example of a function f so that* $f'(x) = x^3$? As the course progresses, we emphasize the chain rule with problems paralleling most texts' sections on indefinite integrals. Consider

$$\int \frac{x}{\sqrt{x+5}} dx.$$

In its place, before introducing the indefinite integral or u-substitution, we would ask, *Can you find an example of a function f so that $f'(x) = \frac{x}{\sqrt{x+5}}$ for $x > -5$?* With students already adept at "guessing" elementary antiderivatives, we take the following approach. Suppose we let

$$u(x) = \sqrt{x+5} \text{ for all } x > -5$$

so that

$$u'(x) = \frac{1}{2\sqrt{x+5}}.$$

Then we have for all $x > -5$,

$$\begin{aligned} f'(x) &= 2u'(x) \cdot x \\ &= 2u'(x)(u^2(x) - 5) \\ &= 2u^2(x)u'(x) - 10u'(x). \end{aligned}$$

If the chain rule has been properly emphasized, students will recognize and suggest that

$$f(x) = \frac{2}{3}u^3(x) - 10u(x) = \frac{2}{3}\left(\sqrt{(x+5)^3} - 10\sqrt{x+5}\right)$$

satisfies the desired property. After a student produces an antiderivative, we will point out that adding any non-zero constant will produce a distinct antiderivative. Once our students are adept at "guessing" antiderivatives by using the chain rule, we familiarize them with traditional notation. Here is an example where we tie our approach in with the traditional notation of the indefinite integral, followed by a traditional approach. Notice how the alternative reinforces the chain rule.

Our approach to evaluate $\displaystyle\int \frac{1}{\sqrt{x+5}}\, dx.$

1. Let $u(x) = \sqrt{x+5}$ for all $x > -5$.
2. Thus $u'(x) = \dfrac{1}{2\sqrt{x+5}}$ and $2u'(x) = \dfrac{1}{\sqrt{x+5}}$ for all $x > -5$.

3. Conclude

$$\int \frac{x}{\sqrt{x+5}}\,dx = 2\int u'(x)\big(u^2(x) - 5\big)\;dx = 2\left(\frac{u^3(x)}{3} - 5u(x)\right) + k$$
$$= 2\left(\frac{1}{3}(x+5)^{\frac{3}{2}} - 5\sqrt{x+5}\right) + k.$$

Traditionally:

1. Let $u = \sqrt{x+5}$.
2. Thus $du = \dfrac{1}{2\sqrt{x+5}}\;dx$ and $2du = \dfrac{1}{\sqrt{x+5}}\;dx$.
3. Conclude $\displaystyle\int \frac{x}{\sqrt{x+5}}\,dx = 2\int (u^2 - 5)\;du$
$$= 2\left(\frac{u^3}{3} - 5u\right) + k = 2\left(\frac{1}{3}(x+5)^{\frac{3}{2}} - 5\sqrt{x+5}\right) + k.$$

Why do we start out intentionally avoiding the well-established notation of the indefinite integral? Because neither of us has seen a satisfactory definition for this notation in a calculus text. Many texts define the indefinite integral, $\int f(x)\,dx$, as the class of all antiderivatives of f with respect to x, but with no mention of domain. At least one text writes, "... one may simply think of $\int f(x)\,dx$ as the antiderivative of $f(x)$ with respect to x," while another writes "$\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$." From this definition, a student might note that

$$\frac{x^2}{2} + 1 = \int x\;dx = \frac{x^2}{2} + 2$$

but of course cannot conclude that 2 = 1. Most texts follow their definition with the standard formula, $\int \frac{1}{x}\;dx = \ln(|x|) + c$. This represents one class of antiderivatives but not *all* antiderivatives of the function $f(x) = 1/x$, for all $x \neq 0$. Either

$$\int \frac{1}{x}\;dx = \ln\;x + c \text{ for } x > 0$$

or

$$\int \frac{1}{x}\;dx = \begin{cases} \ln x + c_1 & \textit{if } x > 0 \\ \ln(-x) + c_2 & \textit{if } x < 0. \end{cases}$$

would be more accurate than what is traditionally written.

Our belief is that each mathematical object should have a correct definition and this definition should be used in a consistent manner throughout the course. We now demonstrate how traditional u-substitution contradicts this premise. Recall the typical definition for an indefinite integral,

Definition 1: *$\int f(x)\,dx$ represents the class of all anti-derivatives of the function f, where dx denotes that the anti-differentiation is to be performed with respect to the variable x.*

This definition forces the variable with which the integration is to be performed to be an independent (dummy) variable. When solving an indefinite integral using a substitution such as $x = \sin(u)$, students are led to a notation that is not consistent with this consequence of Definition 1. To see this, consider the traditional approach to the problem

$$\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

Traditionally:

1. Let $x = \sin(u)$ so that $dx = \cos(u)\,du$.
2. Hence, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos(u)}{\cos(u)}\,du = \int 1\,du \ldots$

When an author writes $x = sin(u)$ in considering $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$ we know that x is the independent variable by definition of the indefinite integral. Therefore u is the dependent variable and the statement "Let $x = \sin(u)$" implicitly defines the function $u(x) = \arcsin(x)$ although it *looks* like one is defining x as a dependent variable (function). The final indefinite integral in line 2 is taken with respect to the function u and such an indefinite integral has not yet been defined.

In our treatment of the problem, we start with the assumption that *there exists a function u so that* $\sin(u(x)) = x$ *for all* $x \in (-1, 1)$. This makes clear that u is the dependent variable and x is the independent variable.
Alternatively:

1. Suppose there is a function u so that $\sin(u(x)) = x$ for all $x \in (-1, 1)$.
2. Differentiating, we have $\cos(u(x))\,u'(x) = 1$.
3. Hence, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos(u(x))}\cos(u(x))u'(x)\,dx = \int u'(x)\,dx \ldots$

Our alternative works equally well for more interesting trigonometric substitutions such as $\int \frac{\sqrt{x^2-1}}{x}\,dx$. For this problem, assume that there is a function u such that for each $x \geq 1$, $x = \sec(u(x))$ so that $\sec^2(u(x)) - 1 = \tan^2(x)$.

This approach results in a solution to the problem that encourages the use of functional notation, reinforces the chain rule, uses the indefinite integral in a way that is consistent with Definition 1, and does not require the student to determine from context the independent and dependent variables. This method allows us to work within the context of the standard notation and also to utilize the power of u-substitution in finding antiderivatives while making explicit the relationship between the dependent and independent variables. The algorithmic procedure of u-substitution where the symbols are used in several contexts is contrary to a consistent exposition of the subject, and easily remedied. Furthermore, doing so reiterates the importance of the chain rule which becomes such a powerful tool in the third semester of the subject. For additional approaches, consider [4, 15, 28].

3.1. Course Evaluation Technique

Will a class conduct itself with the professor in absentia? One author was a graduate student at The University of North Texas when his advisor asked him to cover a class while he was away at a conference. When asked what was to be covered, the advisor said, "They'll cover the material, you just watch and see that it is correct." Upon watching the class carefully present problems without their instructor present and with minimal input from the official observer, the author began to modify his own courses. One author now measures the success of the culture of a particular class by whether the class will conduct itself in his absence. He will request an e-mail summary of the class period from each student.

3.2. Course Evaluation Comments

In response to the phrase, "I believe the most effective part of the course was," one student wrote: "The ability for the students to interact in class. I learn better when I get involved." In free response, one student wrote, "Despite the confusing language of the text, the professor was very apt at explaining concepts."

4. CONCLUSION

It is the belief of the authors that the reform of calculus should serve to help students learn to speak correctly about mathematical objects and to encourage them to take more mathematics courses. We feel that simple and precise definitions, notation, and applications are one key component to attracting minors and majors, preparing non-majors, and providing the "training in thinking," advocated by the quotes in the introduction.

Over the years many departments have introduced "bridge" or "transition" courses between calculus and the advanced courses that constitute the bulk of the major. Such courses are intended to train students to use definitions, to read, to understand, and to write correct mathematics. Given that the typical calculus courses constitute between nine and twelve semester hours of the mathematics taken by students majoring in mathematics, the sciences, or engineering, we think that better use of language in calculus courses would facilitate the transition to higher mathematics since students would already have been introduced to carefully written and spoken mathematics. Indeed, traditional calculus courses are potentially an impediment rather than a bridge to higher mathematics.

The decline in the number of majors and the need for transitional courses seem to imply that we are not providing our calculus students with an indication of what mathematics is really about. If true, this is unfortunate and unnecessary. We do not propose that calculus students should be made to produce $\epsilon - \delta$ proofs, but they can be trained to read, interpret, and apply accurately written definitions. This is little different from proving theorems in later courses.

Introducing our subject using the new materials and methods of the reform movement alongside the precise notation and definitions advocated in this paper should provide a powerful ally in preparing and recruiting students. Perhaps a reform theme we can all agree on would be to present the subject in the way in which we view it—as a beautiful, simple, precise and understandable one which provides a stepping stone into numerous fields of advanced study.

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