SINGLE-ITERATION SOBOLEV DESCENT FOR LINEAR INITIAL VALUE PROBLEMS

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Abstract. Sobolev descent has long been established as an efficient method for numerically solving boundary value problems, ordinary differential equations and partial differential equations in a small number of iterations. We demonstrate that for any linear ordinary differential equation with initial value conditions sufficient to assure a unique solution, there exists a Hilbert space in which gradient descent will converge to the solution in one iteration. We provide two elementary examples, one initial value problem and one boundary value problem, demonstrating the effectiveness of the theory in numerical settings. As there are ample efficient numerical methods for solving such problems, the significance of the paper is in the approach and the question it raises. Namely, do such spaces exist for wider classes of differential equations?

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1. Introduction

A general discussion of optimization techniques is given in [9] and [19]. Applying steepest descent to solve differential equations was first introduced by Cauchy in [3] and modifications such as conjugate gradient and variable metric methods were later introduced in [6] to speed up the convergence of numerical implementations. Sobolev descent is a systematic preconditioning technique where gradients are based on Sobolev spaces determined by the problem at hand rather than on Euclidean space. The method was introduced by Neuberger in [14]. Sufficient conditions for convergence and a complete discussion of Sobolev descent may be found in [15]. The extension of the technique to singular ordinary differential equations by utilizing weighted Sobolev spaces based on the problem at hand, [11], extends this work. Problem specific applications of Sobolev descent are given in [4], [5], [7], [12], [13], [18], [20] and [21]. In [10] a convergence proof is given for discrete spaces similar to those in this paper. Existence and uniqueness arguments for singular problems in Sobolev spaces are given in [2] and [22].

The only example to date of a differential equation and a space in which convergence occurs in one iteration is the minimal surface problem, [17]. Yet in each of the papers cited above, the choice of the space is integral to the success of the method. This fact is illustrated both by the well documented failure of descent methods based on the Euclidean metric and the minimal number of iterations required when application specific problems are solved in carefully constructed spaces as in the references above. Thus the efficiency of the descent process, as measured in terms of the number of iterations, is clearly dependent on the choice of the space. Is there an optimal space in which to perform descent? We prove that for linear initial value problems the answer is "yes" by creating the space and proving that convergence occurs in one iteration. We demonstrate that the approach is computationally effective by numerically solving one simple IVP. We then apply the numerical method to one simple BVP (Legendre's) to demonstrate that the numerical algorithm may also be applied effectively to BVPs. MATLAB code is provided for the IVP and available upon request for the BVP.

General references for Sobolev spaces are [1] and [8].

2. Continuous Descent Space

Let $I = [0, 1]$. Fix $n \in \mathbb{N}$. Let $H = H_I^{n,2}$ and $L = L_I^2$. Define $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to be the inner product and norm, respectively, on L. Define $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|_H$ to be the inner product and norm, respectively, on H. Fix $p_0, p_1, ..., p_{n-1} \in C_I$ and define $D : H \to L$ so that for each $y \in H$,

$$Dy = y^{(n)} + p_{n-1}y^{(n-1)} + ... + p_0 y.$$

For each $k = 1, 2, ..., n$, define $B_k : H \to L$ such that $B_k y = y^{(k-1)}(0)$ for every $y \in H$. Define $K : H \to L^{n+1}$ by

$$Ky = (Dy, B_1 y, ..., B_n y) \quad \forall\ y \in H.$$

Let $X = H$. Define $\langle \cdot, \cdot \rangle_X : X \times X \to \mathbb{R}$ by

$$\langle y, z \rangle_X = \langle Ky, Kz \rangle_{L^{n+1}} \quad \forall\ y, z \in X.$$

Define $\|\cdot\|_X : X \to \mathbb{R}^{\geq 0}$ by

$$\|y\|_X = \sqrt{\langle y, y \rangle_X} \quad \forall\ y \in X.$$

We will show that $(X, \langle \cdot, \cdot \rangle_X)$ defines a Hilbert space and that for any given $f \in L$ and $g \in H$, steepest descent based on the gradient induced by this inner product and applied to $\phi : X \to \mathbb{R}$ where $\phi(y) = \frac{1}{2} \|Dy - f\|^2$ will converge in one iteration to the unique solution of

$$Dy = f; \quad B_k y = B_k g,\ \forall\ k = 1, ..., n.$$

The heart of the paper is that once we have defined the gradient $\nabla_X \phi$ based on our space $(X, \langle \cdot, \cdot \rangle_X)$ we will have for any $y \in X$, that $Dy = D\nabla_X \phi(y)$. From this it follows immediately that $\phi(y - \nabla_X \phi(y)) = \frac{1}{2}\|D(y - \nabla_X \phi(y))\|^2 = \frac{1}{2}\|Dy - D\nabla_X \phi(y)\|^2 = 0$ and thus convergence to a solution is guaranteed in one iteration.

Theorem 2.1. *$(X, \langle \cdot, \cdot \rangle_X)$ defines a Hilbert space.*

Proof. It is easily shown that $\langle \cdot, \cdot \rangle_X$ defines an inner product on X. In order to show that $(X, \langle \cdot, \cdot \rangle_X)$ is complete, fix $y_1, y_2, ...$ to be Cauchy under $\|\cdot\|_X$. It follows that $\|Dy_i - Dy_j\| \to 0$ as $i, j \to \infty$. Since L is complete, there exists $f \in L$ so that $Dy_i \to_L f$ as $i \to \infty$. If $k \in \{1, 2, \ldots, n\}$ then the map $B_k : H \to L$ maps every function to a constant function. Therefore, $\|B_k y_i - B_k y_j\| \to 0$ as $i, j \to \infty$ implies that $B_k y_1, B_k y_2, ...$ is Cauchy in $\mathbb{R}$. For each $k \in \{1, 2, \ldots, n\}$ let $b_k \in \mathbb{R}$ so that $B_k u_i \to b_k$ as $i \to \infty$. Defining $y \in X$ so that $Ky = (f, b_1, ..., b_n)$ we have

$$\lim_{i\to\infty} \|y - y_i\|_X = \lim_{i\to\infty} \|Ky - Ky_i\|_{L^{n+1}} = \lim_{i\to\infty} \Big[\|f - Dy_i\| + \sum_{k=1}^{n} \|b_k - B_k y_i\| \Big] = 0.$$

Since $y \in X$, we have that X is complete under $\langle \cdot, \cdot \rangle_X$. □

Lemmas 2.2, 2.3 and 2.4 define a projection which plays a crucial role in defining the gradient upon which the descent process is based. Define $Q = \{Ky : y \in X\} \subseteq L^{n+1}$ and let $Q^\perp$ denote the orthogonal complement of Q in L^{n+1}. Let

$$M = \left\{ (0, v_1, ...v_n) : v_k \in C_I \text{ and } \int_I v_k = 0 \, \forall \, k = 1, 2, ..., n \right\}.$$

Lemma 2.2. *$M \subseteq Q^\perp$.*

Proof. Fix $p = (g, b_1, ..., b_n) \in Q$ and $q = (0, v_1, ..., v_n) \in M$. Then

$$\langle p, q \rangle_{L^{n+1}} = \langle g, 0 \rangle + \sum_{k=1}^{n} \langle b_k, v_k \rangle = \sum_{k=1}^{n} \int_0^1 b_k v_k = \sum_{k=1}^{n} b_k \int_0^1 v_k = \sum_{k=1}^{n} b_k \cdot 0 = 0.$$

□

Lemma 2.3. *For every $f \in L \times (C_I)^n$ there exists a unique pair $(y, v) \in X \times M$ so that $Ky + v = f$.*

Proof. Fix $f = (f_0, f_1, ..., f_n) \in L \times (C_I)^n$. For each $k = 1, 2, \ldots n$ let $v_k = f_k - \int_I f_k$ and $b_k = \int_I f_k$. Let y satisfy $Ky = (f_0, b_1, ..., b_n)$. Thus $Ky + v = (f_0, b_1, \ldots, b_n) + (0, v_1, \ldots, v_n) = (f_0, b_1 + v_1, \ldots, b_n + v_n)$. Since for each $k = 1, 2, \ldots n$ we have that $b_k + v_k = \int_i f_k + (f_k - \int_i f_k) = f_k$, we have $Ky + v = f$. If $(y, v), (z, w) \in X \times M$ satisfy $Ky + v = f$ and $Kz + w = f$ then $K(y - z) = -(v - w)$ so $\big(D(y - z), (y - z)(0), \ldots (y - z)^{(n)}(0)\big) = (0, v_1 - w_1, \ldots, v_n - w_n)$ from which it follows that $y = z$ and $v = w$. □

Define $P : L \times (C_I)^n \to Q$ so that for each $f \in (C_I)^{n+1}$, $Pf = Ky$ where y is guaranteed by the proof of the previous theorem. Since Q and M are mutually orthogonal and $Q \oplus M$ is dense in L^{n+1}, then $Q \oplus Q^\perp = L^{n+1}$. Thus, we may extend P by continuity so that $P : L^{n+1} \to Q$.

Lemma 2.4. *$P^2 = P$ and $\langle Pf, g \rangle_{L^{n+1}} = \langle f, Pg \rangle_{L^{n+1}}$ for every $f, g \in L^{n+1}$.*

Proof. Fix $f \in L^{n+1}$. Let $y \in X$ be the unique element which satisfies $Pf = Ky$. Let $z \in X$ be the unique element which satisfies $P(Ky) = Kz$. By definition of P, we may define $v \in M$ to be the unique element so that $Kz + v = Ky$. Since $z = y$ and $v = 0$ satisfy $Kz + v = Ky$, then z must be y and we have that $Kz = Ky$, hence $P^2 f = Pf$. Therefore $P^2 = P$.

Fix $f = (f_0, f_1, ..., f_n)$ and $g = (g_0, g_1, ..., g_n)$ to be elements of L^{n+1}. First note that for any k,

$$\begin{aligned}\left\langle \int_I f_k, g_k \right\rangle &= \int_I \left(\int_I f_k \right) g_k = \left(\int_I f_k \right) \int_I g_k \\ &= \left(\int_I g_k \right) \int_I f_k = \int_I \left(\int_I g_k \right) f_k = \left\langle f_k, \int_I g_k \right\rangle .\end{aligned}$$

Then

$$\begin{aligned}\langle Pf, g \rangle_{L^{n+1}} &= \langle f_0, g_0 \rangle + \sum_{k=1}^{n} \left\langle \int_I f, g \right\rangle \\ &= \langle g_0, f_0 \rangle + \sum_{k=1}^{n} \left\langle f_k, \int_I g_k \right\rangle = \langle f, Pg \rangle_{L^{n+1}} .\end{aligned}$$

□

Define the function $\pi : Q \to X$ so that $\pi(Ky) = y$ for every $y \in X$. Fix $f \in L$ and $b_1, ..., b_n \in \mathbb{R}$. Define $\phi : X \to \mathbb{R}$ by $\phi(y) = \frac{1}{2} \|Dy - f\|^2$ for every $y \in X$. Since X and L are Hilbert spaces, for each $y \in X$, the map $\phi'(y)$ is a bounded linear functional on both $(L, \langle \cdot, \cdot \rangle)$ and $(X, \langle \cdot, \cdot \rangle_X)$. Hence, we may define $\nabla \phi(y)$ and $\nabla_X \phi(y)$ so that for every $h \in X$ we have

$$\langle \nabla \phi(y), h \rangle = \phi'(y)h = \langle \nabla_X \phi(y), h \rangle_X .$$

We now show that gradient descent within X preserves the initial conditions and produces, in a single iteration, a zero of ϕ, yielding a solution to $Ky = (f, b_1, ..., b_n)$.

Theorem 2.5. *For every $y \in X$, the function $u = y - \nabla_X \phi(y)$ satisfies $\phi(u) = 0$ and $B_k u = B_k y$, $k = 1, ..., n$.*

Proof. Fix $y \in X$. Then for any $h \in X$ we have

$$\begin{aligned}\phi'(y)(h) &= \langle Dy - f, Dh \rangle \\ &= \langle (Dy - f, 0, ..., 0), Kh \rangle_{L_2^{n+1}} \\ &= \langle (Dy - f, 0, ..., 0), P(Kh) \rangle_{L_2^{n+1}} \\ &= \langle P(Dy - f, 0, ..., 0), Kh \rangle_{L_2^{n+1}} \\ &= \langle \pi P(Dy - f, 0, ..., 0), \pi(Kh) \rangle_X \\ &= \langle \pi P(Dy - f, 0, ..., 0), h \rangle_X\end{aligned}$$

Thus, $\nabla_X \phi(y) = \pi P\left(Dy - f, 0, ..., 0\right)$. Define $z \in X$ so that $Kz = P\left(Dy - f, 0, ..., 0\right)$. Then $Dz = Dy - f$ and $B_k z = 0$ for every $k \in \{1, 2, \ldots, n\}$. Moreover, $z = \pi(Kz) = \nabla_X \phi(y)$. Define $u = y - z$. Then

$$\phi(u) = \phi(y-z) = \frac{1}{2}\left\|D(y-z) - f\right\|^2 = \frac{1}{2}\left\|Dy - Dz - f\right\|^2 = \frac{1}{2}\left\|Dy - (Dy - f) - f\right\|^2 = 0$$

and

$$B_k u = B_k(y - z) = B_k y - B_k z = B_k y - 0 = B_k y, \quad \forall\, k = 1, ..., n.$$

□

3. Discrete Example

Discrete Sobolev descent is detailed for general systems in [12] and we give only a brief treatment here to demonstrate the single-step convergence. As there are ample methods for solving linear initial value problems, the significance of the paper lies in the existence of an inner product space where convergence occurs in a single iteration.

Consider $y'' + y = 0$ on $[a, b] = [0, 2\pi]$ with $y(0) = 0$ and $y'(0) = 2$. Let N be the number of points in our partition of $[a, b]$ and $\delta = \frac{b-a}{N-1}$. We first define the discrete version of our differential operator $Du = (D_2 + D_0)u = u'' + u$ in the usual way. Let D_0 and $D_2 \in L(\mathbb{R}^N, \mathbb{R}^{N-2})$ satisfying

$$D_0(x) = \begin{pmatrix} \frac{x_1 + 2x_2 + x_3}{4} \\ \vdots \\ \frac{x_{N-2} + 2x_{N-1} + x_N}{4} \end{pmatrix} \quad \text{and} \quad D_2(x) = \begin{pmatrix} \frac{x_1 - 2x_2 + x_3}{\delta^2} \\ \vdots \\ \frac{x_{N-2} - 2x_{N-1} + x_N}{\delta^2} \end{pmatrix}.$$

Next, we define the discrete version of our initial conditions, $Hu = (u(0), u'(0))$. Let H_0 and $H_1 \in L(\mathbb{R}^N, \mathbb{R}^{N-2})$ such that

$$H_0(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_1 \end{pmatrix} \quad \text{and} \quad H_1(x) = \begin{pmatrix} \frac{x_2 - x_1}{\delta} \\ \vdots \\ \frac{x_2 - x_1}{\delta} \end{pmatrix}.$$

Let $A = D^t D + H_0^t H_0 + H_1^t H_1$. Let $\langle \cdot, \cdot \rangle$ represent the usual dot product and $\langle x, y \rangle_X = \langle Dx, Dy \rangle + \langle H_0 x, H_0 y \rangle + \langle H_1 x, H_1 y \rangle$. Then for all $x, y \in \mathbb{R}^N$ we have $\langle Ax, y \rangle = \langle x, y \rangle_X$. The discrete version of the operator ϕ from the previous section is given by $\phi(y) = \frac{\delta}{2}\|Dy\|^2 = \frac{\delta}{2}\sum_{k=1}^{N-2}(Dy)_k^2$. Applying the Riesz Representation Theorem in these two inner product spaces we see that:

(1) For all $x, y \in X$, $\langle \nabla\phi(x), y \rangle = \phi'(x)(y) = \delta \langle Dx, Dy \rangle = \delta \langle D^t Dx, y \rangle$. Hence, $\nabla\phi(y) = \delta D^t Dy$.
(2) For all $x, y \in X$, $\langle \nabla\phi(x), y \rangle = \phi'(x)(y) = \delta \langle \nabla_X \phi(x), y \rangle_X = \delta \langle A\nabla_X \phi(x), y \rangle$.

Therefore $\nabla_X \phi$ and $\nabla\phi$ are related by $A\nabla_X \phi(x) = \nabla\phi(x)$ for any $x \in X$.

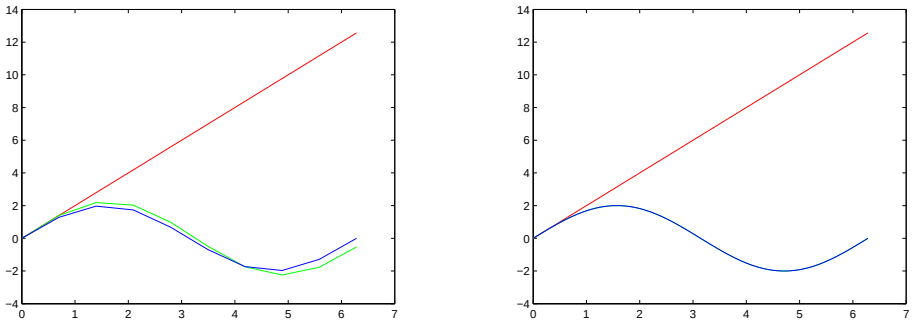

FIGURE 1. $y'' + y = 0$, $y(0) = 0$, $y'(0) = 2$, $N = 10$ and $N = 1000$

This simple application of the Riesz Representation Theorem allows us to compute $\nabla_X \phi(y) = \pi P\left(Dy, 0, ..., 0\right)$ by solving a sparse linear system without explicitly producing either π or P. In the case where the partition of the interval has N elements and D has order n, A is an $N \times N$ matrix with $2n + 1$ non-zero diagonals.

Algorithm

(1) Choose an initial vector y satisfying the initial conditions.
(2) Define D for the differential equation.
(3) Define $H_k,\ k = 1, 2, ..., n-1$ for the initial conditions.
(4) Compute the matrix, $A = D^t D + H_0^t H_0 + H_1^t H_1 + ... + H_{n-1}^t H_{n-1}$.
(5) Compute $\nabla \phi(y) = D^t D y$.
(6) Solve $A \nabla_X \phi(y) = \nabla \phi(y)$ for $\nabla_X \phi(y)$.
(7) Now $s = y - \nabla_X \phi(y)$ is the (single-iteration) solution.

The graphs in Figure 1 show the initial function $y_0(t) = 2t$ on $[0, 2\pi]$ and the resulting solution after one iteration, $y_1(t) = 2\sin(t)$. We demonstrate the result using 10 divisions and 1000 divisions to demonstrate one of the more powerful features of the method. When solving more difficult problems, for example partial differential equations, it is significant that the algorithm provides reasonable precision on a very small number of divisions. The graphs show the initial estimate (the straight line), the approximate solution and the true solution, although on the second graph the approximate and true solution are indistinguishable.

Assuming y is our approximate solution and z is the true solution, Table 1 lists:

(1) $\phi(y) = \frac{\delta}{2} \sum_{k=1}^{N-2} (Dy)_k^2$,
(2) the average absolute error, $\frac{1}{N} \sum_{k=1}^{N} |z_k - y_k|$,
(3) the maximum absolute error, $\max\left\{|z_k - y_k| : k = 1, 2, \ldots, N-2\right\}$,
(4) the average divided difference error $\frac{1}{N-1} \sum_{k=1}^{N-2} |(Dy)_k|$, and
(5) the maximum divided difference error $\max\left\{|(Dy)_k| : k = 1, 2, \ldots, N-2\right\}$.

TABLE 1. Numerical Results: $u'' + u = 0, u(0) = 0, u'(0) = 2$

$y'' + y = 0$		$y(0) = 0$		$y'(0) = 2$	
Divisions	$\phi(y)$	Avg Abs Err	Max Abs Err	Avg Div Err	Max Div Err
$N = 10$	1.8×10^{-28}	2.4×10^{-1}	5.3×10^{-1}	5.6×10^{-15}	1.6×10^{-14}
$N = 1000$	1.1×10^{-14}	2.0×10^{-5}	4.4×10^{-5}	4.4×10^{-8}	1.2×10^{-7}

TABLE 2. Numerical Results: $u' - v = 0, v' + u = 0, u(0) = 0, v(0) = 2$

$u' - v = 0$		$v' + u = 0$	$u(0) = 0$	$v(0) = 2$	
Divisions	$\psi(u, v)$	Avg Abs Err	Max Abs Err	Avg Div Err	Max Div Err
$N = 10$	9.1×10^{-30}	3.9×10^{-3}	6.2×10^{-3}	3.6×10^{-15}	1.1×10^{-14}
$N = 1000$	1.6×10^{-24}	3.2×10^{-7}	5.0×10^{-7}	1.3×10^{-12}	5.5×10^{-12}

While the errors in Table 1 grow as the number of divisions grows, this feature is managed more efficiently by solving the second order equation as the standard first order system,

$$u'(t) - v(t) = 0, v'(t) + u(t) = 0, u(0) = 0, \text{ and } v(0) = 2,$$

and minimizing the corresponding function $\psi(u, v) = \frac{1}{2}\left(\|u' - v\|^2 + \|v' + u\|^2\right)$. The numerical results for the system are given in Table 2. A list of other experiments is given in Section 5.

4. An Elementary Boundary Value Problem

While we do not develop theory for boundary value problems, we demonstrate a parallel numerical approach which may be easily implemented for a wide variety of boundary value problems. We consider Legendre's equation

$$((1 - t^2)u')' + 2u = 0$$

on $[0, 1]$ with boundary conditions $u(0) = 0$ and $u(1) = 1$, and $u \in C_I^2$. General solutions are of the form $u(t) = c_1 t + \frac{c_2}{2}\ t\ ln(\frac{1+t}{1-t})$ for some $c_1, c_2 \in \mathbb{R}$ and only $u(t) = t$ satisfies the given boundary conditions. The finite difference approximation of Legendre's equation is defined for the expanded form $(1 - t^2)u'' - 2tu' + 2u = 0$. Let N be the number of points in the uniform partition of $[0, 1]$ and let $\delta = \frac{1}{N-1}$. Define D_0 as in Section 3 and define the weighted operators $D_1, D_2 \in L(\mathbb{R}^N, \mathbb{R}^{N-2})$ so that

$$D_1(x) = \begin{pmatrix} \frac{\delta(-x_1+x_3)}{2\delta} \\ \vdots \\ \frac{(N-1)\delta(-x_{N-2}+x_N)}{2\delta} \end{pmatrix} \quad \text{and} \quad D_2(x) = \begin{pmatrix} \frac{(1-\delta^2)(x_1-2x_2+x_3)}{\delta^2} \\ \vdots \\ \frac{(1-((N-1)\delta)^2)(x_{N-2}-2x_{N-1}+x_N)}{\delta^2} \end{pmatrix}.$$

TABLE 3. Numerical Results: Legendre's

$((1-t^2)u')' + 2u = 0$		$u(0) = 0$		$u(1) = 1$	
Divisions	$\phi(y)$	Avg Abs Err	Max Abs Err	Avg Div Err	Max Div Err
$N = 10$	2.5×10^{-27}	5.2×10^{-14}	7.2×10^{-14}	6.5×10^{-14}	1.1×10^{-13}
$N = 1000$	1.9×10^{-13}	6.2×10^{-7}	1.0×10^{-6}	4.9×10^{-7}	9.9×10^{-7}

Let $D = D_2 - 2D_1 + 2D_0$. Define H_0 as in Section 3 and define $H_1 \in L(\mathbb{R}^N, \mathbb{R}^{N-2})$ so that

$$H_1(x) = \begin{pmatrix} \frac{x_N}{\delta} \\ \vdots \\ \frac{x_N}{\delta} \end{pmatrix}$$

Let $A = D^t D + H_0^t H_0 + H_1^t H_1$. The descent algorithm parallels the algorithm for the IVP presented in Section 3, thus is not repeated here. Table 3 shows the results obtained in one step starting with the initial function, $y = \sin(t)/sin(1)$. As with the second order IVP, variational techniques as developed in [12] will resolve the increase in error associated with the second-order divided differences.

5. Conclusions and Future Work

While the inner product is dependent on the uniqueness of solutions, even when solving problems without initial conditions, numerical experiments are successful in finding a solution. Thus, even in the case where the space is constructed only based on a semi-inner product, the numerics still work.

Numerical experiments by the second author indicate that this process may be extended to a large class of differential equations under necessary and sufficient supplementary conditions. The following is a list of such problems where machine precision was obtained after a single iteration.

(1) Stieltjes integral condition: Fix $c \in \mathbb{R}$ and define $g : [0,1] \to \mathbb{R}$ to be strictly increasing. The test problem is $y' = y$ on the interval $[0,1]$ under the restriction $\int_0^1 y\, dg = c$.
(2) Singular ODEs: Fix $c \in \mathbb{R}$. The test problem is $ty' = y$, $y(1) = c$, on the interval $[0,1]$.
(3) Laplace's equation: The test problem is $u_{11} + u_{22} = 0$ on the domain $[0,1]^2$ under Dirichlet conditions restricting that the function satisfy given function values on the boundary of $[0,1]^2$.

6. Appendix: MATLAB Code

```
% Solves y''+ y = 0, y(0) = 0, y'(0)=2
function yppy
n = 10;
```

```
a = 0;
b = 2*pi;
del = (b-a)/(n-1);
fprintf('Start program\n')
fprintf('n = %d,   a = %g,   b = %g \n',n,a,b)
t = linspace(a,b,n)';
y = ones(size(t));
y = 2*t;
D0 = zeros(n-2,n);
D2 = zeros(n-2,n);
tmp0 = [1, 2, 1] / 4;
tmp2 = [1, -2, 1] / del / del;
for k=1:n-2
    D0(k, k:k+2) = tmp0;
    D2(k, k:k+2) = tmp2;
end
D = D2 + D0;
H0 = zeros(n-2,n);
H0(:,1) = 1;
H1 = zeros(n-2,n);
tmp1 = [-1,1]/del;
for k=1:n-2
    H1(k, 1:2) = tmp1;
end
B=D'*D;
A = B + H0'*H0 + H1'*H1;
Euc_grad = B*y;
Sob_grad = linsolve(A, Euc_grad);
s = y - Sob_grad;
soln = zeros(size(t));
soln = 2*sin(t);
phi_err = 1/2*sum((D*s).^2)*del;
fprintf('norm of Ds = %g \n', phi_err);
avg_abs_err = sum( abs( soln-s ) )/n;
fprintf('average absolute error = %g \n', avg_abs_err);
max_abs_err = max( abs( soln-s ) );
fprintf('max absolute error = %g \n', max_abs_err);
avg_div_diff_err = sum( abs( D*s ) )/(n-1);
fprintf('avg divided difference error = %g \n', avg_div_diff_err);
max_div_diff_err = max( abs( D*s ) );
fprintf('max divided difference error = %g \n', max_div_diff_err);
plot(t,y,'r',  t,s,'g', t, soln,'b')  % plot initial, approx, true
fprintf('end program \n\n');
end
```

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